A Solution Method for Regular Optimal Control Problems with State Constraints

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Abstract. A method of region analysis is developed for solving a class of optimal control problems with one state and one control variable, including state and control constraints. The performance index is strictly convex with respect to the control variable, while this variable appears only linearly in the state equation. The convexity or linearity assumption of the performance index or the state equation with respect to the state variable is not required.

Key Words. Optimal control, regular problems, state constraints, method of region analysis.

1. Introduction

An optimal control problem of Lagrange form is called regular if the Pontryagin function belonging to it is strictly convex with respect to the control variables. Let the control variables appear linearly in the state equation; then, the control problem is regular if the performance index is strictly convex with respect to the control variables. For the sake of simplicity, let us confine the investigation to the case with one state and one control variable. Still, the state and control constraints considered here cause nontrivial difficulties in solving such problems. In particular, if the convexity of the performance index or the linearity of the state equation with respect to the state variable is not assumed, we cannot apply our present knowledge on linear-quadratic or convex problems, which is given in many publications (for example, see Refs. 1–3).

The basis for our investigation is Pontryagin’s maximum principle as represented in Ref. 4. This is a necessary optimality condition which provides

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the local behavior of optimal solutions, resulting in a series of two-point boundary-value problems. In consequence of the state constraints, the possible number of boundary-value problems is very large or may even be infinite, and the location of the boundary points is generally unknown.

In order to overcome the above difficulties in solving optimal control problems with state constraints, we developed a method of region analysis whose procedure depends on the concrete problem class. This method is discussed in Refs. 5-10 for problems in which the control variables appear linearly and in Refs. 11-13 for two classes of regular control problems with simple state equations and state bounds which are constant.

In this paper, a general class of regular control problems with one state and one control variable is considered. Here, the state equation is linear with respect to the control variable, but there is no a priori requirement with respect to the state variable. Moreover, the state bound may be variable.

2. Problem Statement

Let us investigate the following optimal control problem in a fixed time interval \([t_0, t_f]\): Minimize the functional

\[
I = \int_{t_0}^{t_f} L(t, x(t), u(t)) \, dt,
\]

under the constraints

\[
\begin{align*}
\dot{x}(t) &= f(t, x(t), u(t)), & f(t, \xi, v) &= f_1(t, \xi) + f_2(t, \xi) v, \\
\beta^- \leq u(t) \leq \beta^+, & \alpha^-(t) \leq x(t) \leq \alpha^+(t), \\
e_0(x(t_0)) = 0, & e_f(x(t_f)) = 0.
\end{align*}
\]

Here, \(x\) is a scalar state function and \(u\) is a scalar control function, while \(\xi\) and \(v\) are state or control variables, respectively. The function \(f_1(\cdot, \xi)\) is continuous, while \(L(\cdot, \cdot, v), f_1(t, \cdot), f_2(\cdot, \cdot), e_0(\cdot), e_f(\cdot)\) are continuously differentiable and \(L(t, \xi, \cdot)\) is twice continuously differentiable for \(t \in [t_0, t_f]\), \(\xi \in [\alpha^-(t), \alpha^+(t)]\), \(v \in [\beta^-, \beta^+]\). Also, \(\alpha^-\) and \(\alpha^+\) are twice continuously differentiable.

For the regularity of the control problem, we assume that

\[
L_{uv}(t, \xi, v) > 0,
\]

for \(t \in [t_0, t_f]\), \(\xi \in [\alpha^-(t), \alpha^+(t)]\), \(v \in [\beta^-, \beta^+]\). (3)

Moreover, let

\[
-\infty < \beta^- < \beta^+ < +\infty, \quad \alpha^-(t) < \alpha^+(t), \quad f_2(t, \xi) > 0,
\]

for \(t \in [t_0, t_f]\), \(\xi \in [\alpha^-(t), \alpha^+(t)]\). (4)