Minimarg and Maximarg Operators

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Abstract. Two operators on the set of $n$-person cooperative games are introduced, the minimarg operator and the maximarg operator. These operators can be seen as dual to each other. Some nice properties of these operators are given, and classes of games for which these operators yield convex (respectively, concave) games are considered. It is shown that, if these operators are applied iteratively on a game, in the limit one will yield a convex game and the other a concave game, and these two games will be dual to each other. Furthermore, it is proved that the convex games are precisely the fixed points of the minimarg operator and that the concave games are precisely the fixed points of the maximarg operator.

Key Words. Convex games, marginal contributions, duality, symmetric games.

1. Introduction

Shapley (Ref. 1) used the marginal contributions of a player in a cooperative game to define a map from the set of $n$-person cooperative games to $R^n$. By identifying in the obvious way the Shapley value of a game with an additive game, the Shapley value can be seen as an operator on the set of cooperative games. The fixed points of this operator are precisely the additive games. The underlying assumption in the definition of this operator is that all possible orders of formations of the grand coalition are equally likely. One of the operators to be defined in this paper will reflect a pessimistic view on part of each coalition with regard to the order of formation of the grand coalition...

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coalition. Each coalition believes that this will happen in a worst possible way for it. The other operator reflects an optimistic view on part of each coalition with regard to the order of formation of the grand coalition. Each coalition believes that this will happen in a best possible way for it. In Section 2, the operators will be defined and several properties of these operators will be investigated. It will be proved that, for some classes of games, one of these operators will yield convex games while the other will yield concave games. Furthermore, it will be shown that applying one of the operators iteratively on any game will yield a convex game in the limit, while doing this with the other operator will yield a concave game. In Section 3, it will be shown that a characterization of convex games as fixed points of one of the operators can be given while concave games can be characterized as the fixed points of the other.

2. Minimarg and Maximarg Operators

An n-person cooperative game is an ordered pair \( \langle N, v \rangle \) where \( N = \{1, 2, \ldots, n\} \) is the finite set of players and \( v \) is the characteristic function which assigns to every subset of \( N \) a real number with the condition that \( v(\emptyset) = 0 \). The set of subsets of \( N \) is denoted by \( 2^N \). A subset of \( N \) is called a coalition. Whenever there can be no cause for confusion, the game \( \langle N, v \rangle \) will be identified with the function \( v \). The set of n-person games will be denoted by \( G^n \).

A game \( v \in G^n \) is said to be superadditive if

\[
v(S \cup T) \geq v(S) + v(T), \quad \text{for all } S, T \in 2^N \text{ with } S \cap T = \emptyset.
\]

A game \( v \) is said to be subadditive if the reverse inequality holds in (1) and additive if equality holds.

A game \( v \) is said to be convex if

\[
v(S \cup T) + v(S \cap T) \geq v(S) + v(T), \quad \text{for all } S, T \in 2^N.
\]

A game \( v \) is said to be concave if the reverse inequality holds in (2).

Let \( v \in G^n \). The dual game \( v^* \) of \( v \) is defined by

\[
v^*(S) = v(N) - v(N \setminus S), \quad \text{for all } S \in 2^N.
\]

Let \( v \in G^n \). The core \( C(v) \) of \( v \) is defined by

\[
C(v) := \left\{ x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S), \text{ for all } S \in 2^N \right\}.
\]