Deterministic Equivalent for a Continuous-Time Linear-Convex Stochastic Control Problem

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Communicated by R. Rishel

Abstract. We consider a finite-horizon control model with additive input. There are two convex functions which describe the running cost and the terminal cost within the system. The cost of input is proportional to the input and can take both positive and negative values. It is shown that there exists a deterministic control problem whose optimal cost is the same as the one in the stochastic control problem. The optimal policy for the stochastic problem consists of keeping the process as close to the optimal deterministic trajectory as possible.

Key Words. Stochastic control problems, linear-convex optimal control problems, Brownian motion, control process of bounded variation, Green's theorem approach.

1. Introduction and Statement of the Problem

We consider a stochastic linear system with additive noise and additive input which is under our control. The controlled process is described by a stochastic differential equation,

\[ dx(t) = ax(t) \, dt + \sigma \, dw(t) + d\nu(t), \quad x(0) = x. \]

Here, \( x(t) \in \mathbb{R}^1 \) represents the coordinate of the system, \( \sigma > 0 \) and \( a \) are constants, \( w(t) \) is a standard Wiener process on \( (\Omega, \mathcal{F}, \mathcal{F}_t, P) \), and \( \nu(t) \) is an \( \mathcal{F}_t \)-adapted process of bounded variation.

\textsuperscript{1}This research is supported by NSERC Grant A4619, MRCO, NSF Grant DMS-86-01510, and AFOSR Grant 87-0278.

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The running cost is described by a function $g(x(t), t)$, and the terminal cost by the function $G(x)$. A constant $c > 0$ represents a unit cost of input. The objective is to find

$$\min \mathbb{E} \left\{ \int_0^T g(x(t), t) \, dt + c \nu(T) + G(x(T)) \right\},$$

(2)

where the minimum is taken over all $\mathcal{F}_t$-adapted processes $\nu$ of finite expected variation. Note that the control cost in (2) depends on the terminal value $\nu(T)$ of the control process and not on the total variation of the control process.

Parallel to the above stochastic problem, we consider a deterministic control problem

$$dy(t) = ay(t) \, dt + dU(t), \quad y(0) = x,$$

(3)

with an objective to find

$$\min_U \left( \int_0^T g(y(t), t) \, dt + cU(T) + G(y(T)) \right).$$

(4)

It will be shown that there exists an optimal path $y^*(\cdot)$ such that, whatever is the initial state, the optimal policy consists of following this path.

In stochastic problems, the optimal behavior looks similar to that in the deterministic case, in the sense that it is necessary to follow $y^*(\cdot)$ as close as possible. The optimal policy, however, in the stochastic case does not exist, because the control which forces a Brownian motion into a deterministic path is of unbounded variation.

We will also consider deterministic and stochastic problems with bounded control rates. In these problems, $U$ is subject to

$$U(t) = \int_0^t u(s) \, ds, \quad |u(s)| \leq M.$$  

(5)

It will be shown that, when $M \to \infty$, the optimal cost in these problems converges to the optimal cost of the original problem.

It is worthwhile to mention that in problems, deterministic or stochastic, in which there are no restrictions on control rates, the classical Bellman equation in the dynamic programming framework is not applicable; see, e.g., Bertsekas (Ref. 1) and Fleming and Rishel (Ref. 2). In our case, the optimal trajectory is found instead by a simple application of Green's theorem to the graphs of the paths of the controlled processes; see also Miele (Ref. 3) and Sethi and Thompson (Ref. 4).

It is interesting to contrast our results with the discrete-time analog of this problem treated in Bes and Sethi (Ref. 5). While in both cases it is possible to obtain equivalent deterministic problems, there are certain important differences between them. In the discrete-time case, the optimal