Differential Stability
in Non-Lipschitzian Optimization

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Abstract. In this paper, upper and lower bounds are established for the Dini directional derivatives of the marginal function of an inequality-constrained mathematical program with right-hand-side perturbations. A nonsmooth analogue of the Cottle constraint qualification is assumed, but the objective and constraint functions are not assumed to be differentiable, convex, or locally Lipschitzian. Our upper bound sharpens previous results from the locally Lipschitzian case by means of a subgradient smaller than the Clarke generalized gradient. Examples demonstrate, however, that a corresponding strengthening of the lower bound is not possible. Corollaries of this work include general criteria for exactness of penalty functions as well as information on the relationship between calmness and other constraint qualifications in nonsmooth optimization.

Key Words. Marginal function, differential stability, generalized gradients, tangent cones, exact penalty functions, constraint qualifications.

1. Introduction

We consider the parametric mathematical program

\[ (P(s)) \quad v(s) := \min \{ f(x) | g_i(x) \leq s_i, \ i \in J, \ x \in C \}, \]

where \( f \) and each \( g_i \) are functions from \( \mathbb{R}^n \) into \( \mathbb{R} \), \( J := \{1, \ldots, m\} \), \( C \) is a nonempty subset of \( \mathbb{R}^n \), and \( s = (s_1, \ldots, s_m) \in \mathbb{R}^m \). Much is known about the directional derivatives of the marginal function \( v \) (see, for example, Ref. 1). In particular, if \( f \) and \( g \) are locally Lipschitzian and a Cottle-type constraint qualification holds at some local minimizer of \( P(0) \), an upper bound can be
derived for the upper Dini directional derivatives of \( v \) at 0 (Ref. 2). If in addition a uniform compactness (Ref. 3) or tameness (Refs. 4, 5) condition holds and the constraint qualification is satisfied at every local minimizer of \( P(0) \), the lower Dini derivatives of \( v \) at 0 can be bounded below (Ref. 2). These bounds involve the support function of the set of Kuhn–Tucker multipliers at a local minimizer of \( P(0) \) for Kuhn–Tucker conditions stated in terms of Clarke generalized gradients (Ref. 6). They are valid for programs that include right-hand-side perturbations of smooth equality constraints in addition to the inequality constraints of \( (P(s)) \).

In this paper, we extend the results of Auslender (Ref. 2) in two directions for the inequality-constrained problem \( (P(s)) \). First, we do not assume that \( f \) and \( g_i \) are locally Lipschitzian; instead, we merely assume that the domains of certain generalized directional derivatives have sufficient intersection. Second, we show that subgradient sets smaller than the Clarke generalized gradient can be used to sharpen the upper bound on the upper Dini derivatives of \( v \). These improvements are made possible by recent results in the calculus of generalized directional derivatives (Refs. 7, 8).

We begin in Section 2 by reviewing some general Kuhn–Tucker type necessary optimality conditions for \( (P(0)) \) and discussing the properties of the associated sets of multipliers. In Section 3, we establish our upper bound for the upper Dini directional derivatives of \( v \) at 0. We present in Section 4 a lower bound for the lower Dini derivatives of \( v \) at 0. Unlike our upper bound, this lower bound seems to be valid only for the multiplier set associated with the Clarke subgradient. In Section 5, we use the existence of this lower bound to deduce a new result on exactness of penalty functions.

To set the stage for our discussion, we pause now to review some necessary definitions and notation. Let \( \| \cdot \| \) be a norm on \( \mathbb{R}^n \). For \( \epsilon > 0 \), define

\[
B(x_0, \epsilon) := \{ x \in \mathbb{R}^n | \| x - x_0 \| \leq \epsilon \}.
\]

For a set \( C \subset \mathbb{R}^n \), we will denote the interior of \( C \) by \( \text{int} C \), the relative interior of \( C \) by \( \text{ri} C \), and the recession cone of \( C \) by \( 0^+ C \) (Ref. 9). The indicator function of \( C \) is the function \( i_C : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) defined by

\[
i_C(x) := \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{else.} \end{cases}
\]

The negative polar cone of a convex cone \( C \subset \mathbb{R}^n \) will be denoted by \( C^0 \).

For a function \( f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm \infty\} \), the domain of \( f \) is the set

\[
dom f := \{ x \in \mathbb{R}^n | f(x) < +\infty \}.
\]