Approaches to Consistency Adjustment

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Abstract. Saaty’s proposals for deriving a consistent ranking from a reciprocal matrix have generated a lively debate. This paper distinguishes between inference and description in consistency adjustment, offers an interpretation and a critique of the eigenvalue method of consistency adjustment, and discusses properties of the methods of weighted least squares and least lines, through which certain types of constraints may easily be imposed on the ranking.

Key Words. Consistency adjustment, eigenvectors, weighted least squares, least lines.

1. Introduction

An expert is asked to estimate the relative wealth of seven nations. He guesses that Japan is twice as wealthy as Germany and that Germany is three times wealthier than France. Therefore, he should estimate the ratio of Japan’s wealth to France’s as six to one. However, he responds inconsistently that the ratio is three to one. In this instance, the inconsistency arises from human error; but anomalies may also occur if the respondent is required to choose his ratios from a bounded set (e.g., from 1/9 to 9).

The wealth estimates involve paired-comparison responses, which are applied by Saaty and Vargas (Refs. 1–4) to construct rankings. For \( n \) alternatives (like \( n = 7 \) nations), the responses are arrayed in a square matrix \( R \) of order \( n \) whose typical element \( r_{ij} > 0 \) estimates the dominance of alternative \( i \) over \( j \). Consistency requires that \( r_{ji} = 1/r_{ij} \); accordingly, \( r_{ii} = 1 \). Consistency further requires that \( r_{ik} = r_{ij}r_{jk} \); but this latter condition may not be met in practice, as the wealth comparisons show.

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Specifically, if the paired comparisons are entirely consistent, then $R$ has rank one, while inconsistencies increase the rank of $R$. Various techniques have been proposed for approximating an inconsistent $R$ matrix by a matrix of rank one. This paper has three objectives: (i) to distinguish between inference and description in consistency adjustment; (ii) to offer an interpretation and a critique of the eigenvalue method of consistency adjustment; and (iii) to demonstrate some desirable properties of the method of weighted least squares, which was introduced in this journal (Ref. 5) and which easily handles additional constraints that may usefully be imposed on the ranking.

2. Criteria for Consistency Adjustment

There are infinitely many rank-one surrogates for an inconsistent $R$ matrix. Any such surrogate is a column vector $v$ of length $n$. In the papers cited above, Saaty and Vargas propose to approximate $R$ by its principal right eigenvector. Under a normalizing constraint, $v$ is the solution to the homogeneous linear equations

$$Rv = \mu v,$$

where $\mu$ is the maximum eigenvalue of $R$. The Perron–Frobenius theorem assures that $\mu$ and $v$ are real, positive, and unique. The rationale for the eigenvalue method (EM) is discussed more fully in Section 3.

Other approaches to consistency adjustment include the least-squares method (LSM, Ref. 6), the chi-square method (CSM, Ref. 7), the logarithmic least-squares method (LLSM, Refs. 3, 4, 6), and the weighted least-squares method (WLSM, Ref. 5). These techniques are defined in Table 1. In addition, one may consider row or column sums of $R$. If the $R$ matrix happens to be consistent, all these approaches yield the same solution.

Throughout the paper, $v$ denotes the rank-one approximation to $R$ for whatever method is being discussed. This convention is intended to simplify notation without causing confusion.

<table>
<thead>
<tr>
<th>Method</th>
<th>Minimand</th>
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<tbody>
<tr>
<td>LSM</td>
<td>$\sum_i \sum_j (r_{ij} - v_i/v_j)^2$</td>
</tr>
<tr>
<td>CSM</td>
<td>$\sum_i \sum_j (r_{ij} - v_i/v_j)^2 (v_j/v_i)$</td>
</tr>
<tr>
<td>LLSM</td>
<td>$\sum_i \sum_j (\log r_{ij} - \log v_i + \log v_j)^2$</td>
</tr>
<tr>
<td>WLSM</td>
<td>$\sum_i \sum_j (r_{ij}v_j - v_i)^2$</td>
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</tbody>
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