Decomposition Algorithm for Convex Differentiable Minimization

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Abstract. In this paper, we propose a decomposition algorithm for convex differentiable minimization. This algorithm at each iteration solves a variational inequality problem obtained by adding to the gradient of the cost function a strongly proximal related function. A line search is then performed in the direction of the solution to this variational inequality (with respect to the original cost). If the constraint set is a Cartesian product of $m$ sets, the variational inequality decomposes into $m$ coupled variational inequalities, which can be solved in either a Jacobi manner or a Gauss-Seidel manner. This algorithm also applies to the minimization of a strongly convex (possibly nondifferentiable) cost subject to linear constraints. As special cases, we obtain the GP-SOR algorithm of Mangasarian and De Leone, a diagonalization algorithm of Feijoo and Meyer, the coordinate descent method, and the dual gradient method. This algorithm is also closely related to a splitting algorithm of Gabay and a gradient projection algorithm of Goldstein and of Levitin-Poljak, and has interesting applications to separable convex programming and to solving traffic assignment problems.

Key Words. Convex programming, decomposition, linear complementarity, variational inequalities.

1. Introduction

In convex differentiable minimization, one frequently encounters problems whose solution simplifies considerably if the cost functions were

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separable. Examples of this include problems whose constraint sets have product forms (such as the traffic assignment problem) or are polyhedral (Ref. 1). In this case, it is desirable to approximate the original cost function by a sequence of separable cost functions. The classical gradient descent method is one example of a method that follows this approach (approximating the original cost by a sequence of linear costs), but it suffers from slow convergence. Another example is the coordinate descent method, but the convergence of this method requires the cost to be in some sense strictly convex and in general applies to only the Gauss-Seidel version. Recently, Feijoo and Meyer (Ref. 2, also see Ref. 3 for the quadratic case) proposed a Jacobi version of the coordinate descent method that circumvents the difficulties with convergence by introducing a line search at each iteration. Also recently, Mangasarian and De Leone (Ref. 4) proposed a matrix splitting method for solving symmetric linear complementarity problems that also introduces a line search step at each iteration. In this paper, we show that these two methods may be viewed naturally as special cases of a Jacobi-type feasible descent method. This Jacobi method, at each iteration, uses as the descent direction the solution to a variational inequality problem obtained by adding to the gradient of the cost function a continuous, strongly "proximal related" function. A line search (possibly inexact) is then performed along this direction. A major advantage of this method is that each strongly proximal related function can be chosen to match either the structure of the constraint set or the structure of the cost function (or both). Furthermore, when the constraint set is a Cartesian product, it can be implemented in a Gauss-Seidel manner, thus accelerating the convergence. A special case of this Gauss-Seidel method is the classical coordinate descent method (see Refs. 5–9). This method can alternatively be implemented as a dual method for minimizing strongly convex (possibly nondifferentiable) functions subject to linear constraints, a special case of which is the dual gradient method (see Section 6 in Ref. 10). It is also closely related to a splitting algorithm of Gabay (Ref. 11) and a gradient projection algorithm of Goldstein (Ref. 12) and of Levitin-Poljak (Ref. 13), the main difference being that an additional line search is used at every iteration.

This paper proceeds as follows. In Section 2, we introduce the notion of strictly and strongly proximal related functions. In Section 3, we describe the Jacobi-type feasible descent method and establish its convergence. In Section 4, we give a Gauss-Seidel version of this method for problems whose constraint set is a Cartesian product. In Section 5, we give dual versions of the previous methods for minimizing strongly convex functions subject to linear constraints. In Section 6, we study the relationship between the new methods and those known and we propose (new) applications to separable cost problems and traffic assignment problems. In Section 7, we discuss extensions.