On a Degenerate Variational Inequality with Neumann Boundary Conditions

J. L. Menaldi

Communicated by R. Rishel

Abstract. A stopping time problem for degenerate reflected diffusions is studied in this paper. We give a characterization of the optimal cost as the maximum solution of a degenerate elliptic variational inequality with Neumann boundary conditions.

Key Words. Variational inequalities, Neumann boundary conditions, elliptic-parabolic operators, stopping time problems, degenerate reflected diffusion.

1. Introduction and Summary of Main Results

This article develops the proofs of results announced in a previous note (Ref. 1).

Bensoussan and Lions (Ref. 2) have introduced the variational inequality approach in order to solve the optimal stopping time problem in the case of nondegenerate diffusions. Robin (Ref. 3) treated the optimal stopping time problem for Feller processes. We also refer to Shiryayer (Ref. 4). Let us mention that, in the books of Friedman (Ref. 5), Fleming and Rishel (Ref. 6), and Oleinik and Radkevic (Ref. 7), similar problems are considered in a different context.

In this paper, using the technique presented in Ref. 8, we develop the case of a degenerate variational inequality with Neumann boundary conditions associated to the optimal stopping time problem for reflected diffusions. We combine analytic and probabilistic methods.

1 The author would like to thank Professor L. C. Evans for very helpful suggestions on this topic.

2 Visiting Assistant Professor, Department of Mathematics, University of Kentucky, Lexington, Kentucky and I.N.R.I.A., France. Presently, Assistant Professor, Department of Mathematics, Wayne State University, Detroit, Michigan.
Let \((\Omega, \mathcal{F}, P)\) be a probability space and \(\{\mathcal{F}_t\}_{t \geq 0}\) be a nondecreasing right-continuous family of complete sub \(\sigma\)-fields of \(\mathcal{F}\).

Now let
\[ y(t) = y_x(t, \omega), \quad t \geq 0, \omega \in \Omega, \]
be the reflected diffusion on a smooth bounded domain \(\bar{\Omega} \subset \mathbb{R}^N\), starting at \(x \in \bar{\Omega}\) with Lipschitz continuous coefficients \(g(\cdot), \sigma(\cdot), \gamma(\cdot)\).

Next, let \(f(\cdot), \psi(\cdot)\) be real bounded measurable functions on \(\bar{\Omega}\), and let \(\theta\) be any stopping time. The cost functional \(J_x(\theta)\) is given by
\[ J_x(\theta) = \mathbb{E}\left\{ \int_0^\theta f(y(t)) \exp(-\alpha t) \, dt + 1_{\theta < \infty}\psi(y(\theta)) \exp(-\alpha \theta) \right\}, \quad (1) \]
where \(\alpha\) is a positive constant.

Our purpose is to characterize the optimal cost
\[ \hat{u}(x) = \inf \{ J_x(\theta) \mid \theta \text{ stopping time} \} \quad (2) \]
and to obtain an optimal stopping time.

We denote by \(A_0\) the second-order differential operator associated to the Itô equation\(^3\)
\[ A_0 = -\frac{1}{2}\text{tr}[\sigma\sigma^*(\partial^2/\partial x^2)] - g(\partial/\partial x) \quad (3) \]
and
\[ A = A_0 + \alpha. \]

We use the following integral formulation of the operators \(A\) and \(\gamma(\partial/\partial x)\) for any real bounded measurable functions on \(\bar{\Omega}\), \(u\) and \(v\):
\[ \tilde{A}u \leq v \text{ in } \bar{\Omega}, \text{ if the process} \]
\[ X_t = \int_0^t v(y(s)) \exp(-\alpha s) \, ds + u(y(t)) \exp(-\alpha t), \quad t \geq 0, \quad (4) \]
is a strong submartingale for each \(x \in \bar{\Omega}\).

Finally, we introduce the following problem: To find a real bounded measurable function \(u(x)\) on \(\bar{\Omega}\), such that
\[ u \leq \psi, \quad \text{in } \bar{\Omega}, \]
\[ \tilde{A}u \leq f, \quad \text{in } \bar{\Omega}. \quad (5) \]

We obtain the following characterization.

---

\(^3\) If \(B\) is a matrix, then \(B^*\) denotes the transpose of \(B\) and \(\text{tr}(B)\) the trace of \(B\).