Optimal Control of a Heat Transfer Problem with Convective Boundary Condition

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Communicated by R. Rishel

Abstract. We consider the problem of controlling the solution of the heat equation with the convective boundary condition taking the heat transfer coefficient as the control. We take as our cost functional the sum of the $L^2$-norms of the control and the difference between the temperature attained and the desired temperature. We establish the existence of solutions of the underlying initial boundary-value problem and of an optimal control that minimizes the cost functional. We derive an optimality system by formally differentiating the cost functional with respect to the control and evaluating the result at an optimal control. We show how the solution depends in a differentiable way on the control using appropriate a priori estimates. We establish existence and uniqueness of the solution of the optimality system, and thus determine the unique optimal control in terms of the solution of the optimality system.

Key Words. Optimal control, heat equation, convective boundary condition, optimality system.

1. Introduction

We consider the problem of controlling the solution of the heat equation with the convective boundary condition taking the heat transfer coefficient as the control. We take as our cost functional the sum of the $L^2$-norms of the control and the difference between the temperature attained and the desired temperature. We establish the existence of solutions of the underlying initial boundary-value problem and of an optimal control that minimizes the cost functional. We derive an optimality system by formally differentiating the cost functional with respect to the control and evaluating the result at an optimal control. We show how the solution depends in a differentiable way on the control using appropriate a priori estimates. We establish existence and uniqueness of the solution of the optimality system, and thus determine the unique optimal control in terms of the solution of the optimality system.

This research was sponsored by the Applied Mathematical Sciences Research Program, Office of Energy Research, U.S. Department of Energy under Contract DE-AC05-84OR21400 with the Martin Marietta Energy Systems. The authors thank David R. Adams for his assistance in clarifying the proof of Proposition 2.1 and appreciate the comments of the referees for needed revisions.

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coefficient as the control. The parabolic partial differential equation for heat
transfer is posed in a bounded, multidimensional domain $\Omega$ for a finite time
interval $0 \leq t \leq t_{\text{max}}$. The boundary condition imposed on $\partial \Omega$ is the convec-
tive boundary condition with ambient temperature zero. The outward
directed normal derivative of the temperature is proportional to the temper-
ature at the boundary (minus the ambient temperature). This is Newton's
law of cooling, the proportionality factor is the heat transfer coefficient, and
it is this function that we take as our control.

The initial boundary-value problem is as follows, with $T(x, t)$ viewed
as temperature at location $x$ and time $t$:

\begin{align}
T_t - \alpha \Delta T &= 0, & \text{in } \Omega \times (0, t_{\text{max}}) = Q, \\
T(x, 0) &= T_{\text{initial}}(x), & \text{in } \Omega, \\
\frac{\partial T}{\partial \eta} &= -hT, & \text{on } \partial \Omega \times (0, t_{\text{max}}).
\end{align}

Here, $\alpha$ is a positive thermophysical constant, $\Omega$ denotes a simple connected
multidimensional domain, $T_{\text{initial}}(x)$ is a nonnegative bounded function,
$\frac{\partial T}{\partial \eta}$ denotes the outward directed directional derivative normal to $\partial \Omega$, and
$h$ is a nonnegative heat transfer coefficient in $L^\infty(\partial \Omega \times (0, t_{\text{max}}))$, which we
will take as our control. Because it is instructive to see how it occurs in the
final control, we choose not to absorb the constant $\alpha$ into a dimensionless
time.

Physically reasonable data for, and physically reasonable solutions of,
heat transfer problems are bounded. It is well known that there exist
functions that are solutions of the heat equation in the interior of domains
with nice boundary that are zero initially and zero on the boundary, but not
zero in the enclosed region; see Widder (Ref. 1). Such solutions are not
bounded, and of course, they are not physically reasonable. It will be
necessary to exclude unbounded solutions of problem (1) from our discus-
sion. This is easily justified. By the maximum principle, it is known that, if
the initial function is nonnegative and bounded, and the heat transfer
coefficient is nonnegative, then any physically reasonable solution of prob-
lem (1) will be nonnegative and not greater than the bounded solution of the
problem obtained from (1) by replacing the Robin boundary condition by a
homogeneous Neumann boundary condition, and this is independent of the
heat transfer coefficient of problem (1). For cooling problems, in a given
domain with no energy sources, the highest possible temperature resulting
from a given initial distribution is attained by insulating the boundary.

For technical reasons associated with some of the proofs to follow, we
define a related dependent variable,

$$u(x, t) = \exp(-\lambda t) T(x, t),$$