Finite Convergence of Algorithms for Nonlinear Programs and Variational Inequalities\textsuperscript{1,2}

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Abstract. Algorithms for nonlinear programming and variational inequality problems are, in general, only guaranteed to converge in the limit to a Karush–Kuhn–Tucker point, in the case of nonlinear programs, or to a solution in the case of variational inequalities. In this paper, we derive sufficient conditions for nonlinear programs with convex feasible sets such that any convergent algorithm can be modified, by adding a convex subproblem with a linear objective function, to guarantee finite convergence in a generalized sense. When the feasible set is polyhedral, the subproblem is a linear program and finite convergence is obtained. Similar results are also developed for variational inequalities.

Key Words. Convergence of algorithms, nonlinear programming, variational inequalities.

1. Introduction

Numerous algorithms have been devised for the solution of the nonlinear programming problem
\[
\text{(NLP) } \min f(x), \quad \text{s.t. } x \in S,
\]
where \( f \) is a continuous function from \( \mathbb{R}^n \) to \( \mathbb{R}^1 \) and \( S \) is a subset of \( \mathbb{R}^n \), and

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for the variational inequality problem which calls for finding a point \( x^* \in S \) such that

\[
\text{(VI)} \quad G(x^*)^T(x - x^*) \geq 0, \quad \text{for all } x \in S,
\]

where \( G \) is a continuous mapping from \( \mathbb{R}^n \) to \( \mathbb{R}^n \). Algorithms for nonlinear programming problems are, in general, only guaranteed to converge in the limit to a Karush–Kuhn–Tucker point (see, e.g., Ref. 1), which may or may not be a local solution. For special cases, in particular for convex programs, convergence to a local solution can be established (see, e.g., Ref. 2). Algorithms for variational inequality problems typically are guaranteed to converge in the limit to a solution if such a solution is unique (see Refs. 3–4). Thus, in most cases, the existing algorithms for problems (NLP) and (VI) converge to either Karush–Kuhn–Tucker points or solution points in an infinite number of steps.

In this paper, we develop conditions on \( f, S \) and the solution point under which any infinitely convergent algorithm can be modified to ensure finite convergence in a generalized sense. The proposed modification consists of imbedding in a given algorithm a subproblem, with a linear objective function and constraints given by \( S \), that should be solved every fixed number of steps. When \( S \) is a convex polyhedron, this modification suffices to guarantee finite convergence to a solution. This idea is reminiscent of the "spacer step" used in nonlinear programming to convert a locally convergent algorithm into a globally convergent one (see, e.g., Ref. 5).

Several authors have developed conditions under which certain algorithms applied to (NLP) with continuously differentiable \( f \) and convex \( S \) have the property, related to finite convergence, that the optimal active constraint set is identified in a finite number of iterations. Bertsekas (Ref. 6) studied the Goldstein–Levitin–Polyak (GLP) gradient projection method (Refs. 7–8) for problem (NLP) with a special rectangular polyhedral set \( S \) and proved that, if the sequence generated by the algorithm converges to a local minimizer which satisfies the linear independence constraint qualification (LICQ), strict complementarity slackness (SCS), and the second-order sufficiency conditions (SOSC), then the set of active constraints is identified in a finite number of steps. Analogous results were proved for the same problem by Bertsekas (Ref. 9) for a projected Newton method and by Gafni and Bertsekas (Ref. 10) for problem (NLP) with general polyhedral set \( S \) and objective \( f \) that has a locally Lipschitz Fréchet derivative \( \nabla f \), when applying a two-metric projection method which is a generalization of the GLP algorithm.

Gafni and Bertsekas assume a strict local minimizer (which is implied by SOSC) as well as conditions that are weakened versions of the LICQ and SCS conditions. A similar result was obtained recently by Dunn (Ref. 11)