Characterization of Constant Policies in Optimal Control

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Abstract. The conditions under which an optimal control problem commonly employed in economics gives rise to a constant optimal control are characterized. The conditions are stated in terms of the properties of the functional form of the integrand of the objective function and of the state equation. These conditions can be checked prior to computation of the optimal control and thereby can simplify its calculation.

Key Words. Optimal control problems, current value Hamiltonian, constant optimal controls, analytic functions, calculus of variations, differential games.

1. Introduction

The focus of most dynamic economic models is analysis of their steady state behavior. The steady state is characterized by constancy of the optimal control as well as of the state variable and the costate variable. The steady state solution is in general not optimal but is approached by the optimal solution in the limit as time approaches infinity. There may be circumstances, however, when the optimal control is constant through time from the outset, rather than just in the limit. Our objective is to characterize the circumstances under which the optimal control is constant through time in terms of the properties of the integrand of the objective function and of the state equation. With these characterizations, it is possible to determine if the optimal control will be constant without actually going through all the computations required to find the optimal control. Moreover, knowledge that the optimal control will be constant simplifies determination of its actual value.

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2. Characterization of Constant Policies in Optimal Control

It is well known that extremals of a calculus of variations problem of the form

$$\max \int_{t_0}^{t_1} f(x'(t)) \, dt,$$

(s.t. $x(t_0) = x_0$, $x(t_1) = x_1$) (1a)

are linear functions of time; see Kamien and Schwartz (Ref. 1). This means that, for the equivalent optimal control problem,

$$\max \int_{t_0}^{t_1} f(u(t)) \, dt,$$

(s.t. $x'(t) = u(t)$, $x(t_0) = x_0$, $x(t_1) = x_1$) (2a)

the optimal control $u^*(t)$ is constant through time, that is, $du^*/dt = 0$. We know, however, that constancy of the optimal control through time is not confined to this case. For example, consider the limit pricing problem:

$$\max \int_{0}^{\infty} \exp(-rt)[R_1(p(t))(1 - F(t)) + R_2(t)F(t)] \, dt,$$

(s.t. $F'(t) = h(p(t))(1 - F(t))$, $F(0) = 0$) (3a)

Here, $r$ is the discount rate; $p(t)$, the control variable, is the price chosen by a monopolistic firm at time $t$; $R_1(p(t))$ is its profit realized prior to rival entry; $R_2(t)$ is its profit realized after rival entry; $F(t)$, the state variable, is the probability of rival entry on or before time $t$; and $h(p(t))$ is the instantaneous conditional probability of rival entry at time $t$, given no prior entry; for this problem, the optimal price is constant through time; see Kamien and Schwartz, pp. 206-208 (Ref. 1).

It is our objective to determine the general circumstances under which the control is constant through time for the class of optimal control problems

$$\max \int_{0}^{\infty} \exp(-rt)f(x(t), u(t)) \, dt,$$

(s.t. $x'(t) = g(x(t), u(t))$, $x(0) = x_0$) (4a)

where $u(t)$ is the control variable and $x(t)$ is the state variable, $x'(t) = dx(t)/dt$. This formulation is commonly employed in intertemporal models in economics and management science. We assume that this problem has a solution and that the necessary conditions are also sufficient. We assume further that $f(x(t), u(t))$ and $g(x(t), u(t))$ are analytic in $u(t)$ and twice