An Approach to Singular Perturbation Problems Insoluble by Asymptotic Methods

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Abstract. Singular perturbation problems not amenable to solution by asymptotic methods require special treatment, such as the method of Carrier and Pearson. Rather than devising special methods for these problems, this paper suggests that there may be a uniform way to solve singular perturbation problems, which may or may not succumb to asymptotic methods. A potential mechanism for doing this is the author's boundary-value technique, a nonasymptotic method, which previously has only been applied to singular perturbation problems that lend themselves to asymptotic techniques. Two problems, claimed by Carrier and Pearson to be insoluble by asymptotic methods, are solved by the boundary-value method.

Key Words. Singular perturbation problems, asymptotic methods, boundary-value techniques.

1. Introduction

Carrier and Pearson (Ref. 1) have noted that there are singular perturbation problems which are not amenable to solution by asymptotic methods. They in turn propose another approach to solve these problems. Their method consists of forming a trial solution consisting of the outer solution plus an inner solution which is scaled by the perturbation parameter $\varepsilon$, raised to an exponent to be determined. Substituting the trial solution in the original differential equation, Carrier and Pearson (by inspecting, noting, or arguing that certain terms must be negligible) evaluate the exponent of $\varepsilon$. Once the exponent is known, they have an approximating solution to the original differential equation. In contrast to asymptotic methods, Carrier and Pearson do not match the inner and outer solutions. Their method is attractive and represents a blend of inner and outer solutions plus skill and insight into simplifying the approximating equations.

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In earlier publications (Refs. 2-4), the author has suggested a non-asymptotic technique (boundary-value method) to solve singular perturbation problems that have been normally attacked by asymptotic methods. This method consists of matching the inner and outer solutions where the inner solution equations are solved as a two-point boundary-value problem with at least one terminal condition supplied by the solution of the outer equation. The point at which the solutions are matched is adjusted so the terminal conditions of the inner solution match the initial conditions of the outer solution.

As part of a continuing effort to ascertain the limits of applicability of the boundary-value technique, we have attempted to solve two of the problems cited by Carrier and Pearson as unsuitable for solution by asymptotic methods. As will be shown, the boundary-value method can solve these two problems with no difficulties.

2. Problem 1

Consider the singular perturbation problem
\[ \varepsilon y''(x) - (2 - x^2)y(x) = -1, \quad -1 \leq x \leq 1, \quad 0 < \varepsilon \ll 1, \]
with the boundary conditions
\[ y(-1) = 0, \quad y(1) = 0. \]
Carrier and Pearson developed the solution
\[ y(x) = \frac{1}{2-x^2} - \exp\left[-\frac{(x+1)}{4\varepsilon}\right] - \exp\left[-\frac{(1-x)}{x^{1/2}}\right]. \]
If in (1) we set \( \varepsilon = 0 \), the outer solution and its derivative are expressed as
\[ y(x) = \frac{1}{2-x^2}, \quad -1 \leq x \leq 1, \]
\[ y'(x) = \frac{2x}{(2-x^2)^2} = 2xy^2(x), \quad -1 \leq x \leq 1. \]
To obtain the inner solution, we set
\[ t = \frac{x}{\varepsilon}, \]
from which it follows that
\[ y(x) = Y(t), \]
\[ y'(x) = (1/\varepsilon) Y'(t), \]
\[ y''(x) = (1/\varepsilon^2) Y''(t). \]
On substituting (6)-(9) into (1), we obtain the inner solution equation
\[ Y''(t) - [2 - (te)^2]eY(t) = -\varepsilon, \quad -1/\varepsilon \leq t \leq t_f, \]