Globally Convergent Algorithm
for Solving Nonlinear Equations

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Abstract. A general iterative method is proposed for finding the maximal root $x_{\text{max}}$ of a one-variable equation in a given interval. The method generates a monotone-decreasing sequence of points converging to $x_{\text{max}}$ or demonstrates the nonexistence of a real root. It is globally convergent. A concrete realization of the general algorithm is also given and is shown to be locally quadratically convergent. Computational experience obtained for eight test problems indicates that the new method is comparable to known methods claiming global convergence.

Key Words. Nonlinear equations, global optimization, global convergence.

1. Introduction

Finding zeros of nonlinear equations efficiently is of major importance and has widespread applications in numerical mathematics. For a good review of the most important algorithms, excellent textbooks are available (Refs. 1 and 2). Fast convergence in the neighborhood of a root and globality of the algorithm are two important criteria for selecting a method for solving a particular equation. Being quadratically convergent, Newton's method is a favorite choice in case a good approximation is at hand, but it may fail to converge if the initial point is far from the root(s). Globalization techniques [e.g., incorporating bisection in the algorithm (Ref. 1) using interval arithmetics (Refs. 3 and 4)] have been proposed to ensure global convergence.

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In this paper, we present an algorithm, based on using a shift parameter in each iteration, which is globally convergent in the strong sense; i.e., it converges to the maximal root starting out of any point lying above it, as opposed to weak globality as defined by Dennis and Schnabel (Ref. 1, p. 5): "...a method that is designed to converge to a system of nonlinear equations from almost any starting point."

Our method, while being globally convergent in the strong sense, is locally quadratically convergent and in certain cases takes larger steps toward the maximal root than Newton's method. It also tells us if there is no root in a given interval. Global information is needed through the estimation of either the first or the second derivative.

First, the method is presented in a general setting and then it is actualized, resulting in an algorithm which is also computationally tested against some known routines.

Extension of the algorithm to the multivariable case is not trivial, but we are hopeful that we can make progress in this direction as well.

2. Algorithm

Consider the equation

\[ f(x) = 0, \]  

where the function \( f: \mathbb{R} \to \mathbb{R} \) is assumed to have the following properties:

(i) it is continuously differentiable on a given closed interval \([0, x_0]\);
(ii) \( f(x_0) > 0 \) and, if (1) has a solution, then its maximal root \( x_{\max} \) is an interior point of \([0, x_0]\).

We assume furthermore that we have a function \( F: \mathbb{R} \to \mathbb{R} \) which is such that:

(a) it is monotonously increasing and continuously differentiable on the real line;
(b) either \( F'(x) \to \infty \) if \( x \to \infty \), or \( F'(x) \to \infty \) if \( x \to -\infty \), or both hold.

Our algorithm for finding an approximate root of (1) is based on the following iterational step. Given a positive real number \( x_0 \), a candidate \( x_1 \) to be the next iterational point is determined by

\[ F(x_1 + d) = F(x_0 + d) - f(x_0), \]  

or equivalently,

\[ x_1 = F^{-1}(F(x_0 + d) - f(x_0)) - d, \]  

where \( d \) is a suitably chosen parameter and \( F^{-1} \) is the inverse of \( F \).