TECHNICAL NOTE

Sensitivity of Sample Values to Parameter Changes

M. A. Eyler

Communicated by Y. C. Ho

Abstract. The effect of small changes in parameter values on the sample values is investigated further. The concept of linearity with respect to parameters is introduced, and the linear and the nonlinear cases are treated separately. The relation with the sample-path analysis is discussed.

Key Words. Sensitivity, sample-path analysis, probability distributions, linearization.

1. Introduction

A fundamental question in sensitivity analysis of probabilistic systems was raised recently by Suri (Ref. 1). The question can be rephrased in a more general context as follows:

Suppose that a random variable $X$ is observed $n$ times with the outcome $x_1, \ldots, x_m$ when the parameters of the system have value $\theta$. What would the sample values be, if the parameters were changed to $\theta + \Delta \theta$?

As an example, $x_i$ can be the $i$th service time of a particular server in a queueing network, where $\theta$ denotes the mean service time of the same server.

To answer this question, Suri makes two assumptions.

(A1) The distribution function of $X$ is known to be $P(X \leq x) = F(x; \theta)$.

(A2) There exists an underlying random variable $U$, uniformly distributed in $[0, 1]$, which determines $X$ by $X = G(U; \theta)$.

It is easy to see that

$$F = G^{-1},$$

1 Associated Professor, Department of Industrial Engineering, Boğaziçi University, Bebek, Istanbul, Turkey.
when $G^{-1}$ exists, since

$$F(x) = P(X \leq x) = P(G(U) \leq x) = P(U \leq G^{-1}(x)) = G^{-1}(x).$$

With these assumptions, the change in the sample values can be calculated as

$$\Delta x_i = G(u_i; \theta + \Delta \theta) - x_i, \quad i = 1, \ldots, n. \quad (1)$$

However, when the underlying values are not known, they have to be computed using the observed sample values:

$$\Delta x_i = G(F(x_i; \theta); \theta + \Delta \theta) - x_i, \quad i = 1, \ldots, n. \quad (2)$$

Equations (1) and (2) represent the general answer to the question stated above. The former is appropriate for simulation studies, whereas the latter will be more useful on real observations (Ref. 1).

In the following sections, these equations will be applied to the special case where $G$ is linear with respect to its parameters. Then, for the nonlinear case, a linearization scheme will be introduced. Finally, the relation of all this to the sample-path analysis will be discussed.

2. Linear Case

Suppose that $G(u; \cdot)$ is a linear function of just two parameters, given by

$$G(u; m, s) = m + s \, g(u), \quad (3)$$

where $g$ is an invertible function of a single variable. An equivalent form would be

$$F(x; m, s) = g^{-1}((x - m) / s), \quad (4)$$

from which it is seen that $m$ is the location parameter and $s$ is the scale parameter of the probability distribution $F$. Note that the sample index $i$ has been dropped, since it has no effect in this discussion.

Equation (2) can now be used to calculate the effect of changes in $m$ and $s$ on $x$:

$$\Delta x = G(F(x; m, s); m + \Delta m, s + \Delta s) - x$$

$$= (m + \Delta m) + (s + \Delta s)g(g^{-1}((x - m) / s)) - x$$

$$= \Delta m + \Delta s(x - m) / s. \quad (5)$$

The same result could be obtained directly from

$$x = m + s \, g(u),$$