TECHNICAL NOTE

Note on Prime Representations of Convex Polyhedral Sets

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Abstract. Consider a convex polyhedral set represented by a system of linear inequalities. A prime representation of the polyhedron is one that contains no redundant constraints. We present a sharp upper bound on the difference between the cardinalities of any two primes.

Key Words. Convex polyhedral sets, linear inequalities, minimal representation, prime representation, redundancy.

1. Introduction

Consider a convex polyhedral set $P$ with initial representation denoted by the augmented matrix $[A|b]$, that is,

$$P = \{ x \in \mathbb{R}^n \mid Ax \leq b; A \in \mathbb{R}^{m \times n} \}.$$

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We say that the representation \([A_1|b_1]\) of \(P\) is a reduction of \([A|b]\) if \([A_1|b_1]\) is obtained from \([A|b]\) by removing at least one redundant constraint (Refs. 1 and 2). The reduction \([A_1|b_1]\) is called a prime (Ref. 3) if it contains no redundant constraints, and is called a minimal prime if it is a prime with minimum cardinality, that is, number of inequalities. We present a sharp upper bound on the difference between the cardinalities of any two primes.

We first note that, if the original representation contains no implicit equalities and no duplicate constraints, then there is a unique prime and it is the minimal representation as defined by Telgen (Ref. 4). Also, if there are no implicit equalities, but there are duplicate constraints, then there is more than one prime, but they are all minimal representations. Finally, if there are implicit equalities, then the primes are not necessarily minimal representations. In fact, in order to obtain a minimal representation, Telgen (Ref. 4) has shown that the implicit equalities must be replaced with explicit equalities.

Since the prime derived by an algorithm depends upon the order in which the constraints are classified, it is possible for primes with different cardinalities to be obtained for the same polyhedral set. The results of this paper can determine whether or not the observed difference is possible, or simply due to implementation error. If the observed difference is correct, the results can be used to provide an upper bound on the dimension of the polyhedral set.

2. Results

Consider the following example. Let

\[
P = \{0\} \subseteq \mathbb{R}^2,
\]

with the original representation

\[
[A|b] = \begin{bmatrix}
1 & 0 & 0 \\
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -1 & 0 \\
1 & -1 & 0 \\
1 & 1 & 0
\end{bmatrix}.
\]