Technical Note

On a Characterization of Clarke's Tangent Cone

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Communicated by F. Giannessi

Abstract. For a subset \( M \) of a normed vector space \( X \), it is shown that, in the characterization given by Elster and Thierfelder of Clarke's tangent cone to \( M \) at a point \( x^0 \in X \), there is a requirement which becomes superfluous when \( x^0 \) belongs to the closure of \( M \).

Key Words. Local conical approximations of sets, Clarke's tangent cone.

1. Introduction

In Refs. 1–4, Elster and Thierfelder introduce a general notion of local cone approximation of a subset \( M \) of a normed vector space \( X \) as a set-valued mapping \( K : 2^X \times X \to 2^X \), associating a cone \( K(M, x^0) \) to each set \( M \) and each point \( x^0 \in X \) in such a way that six axioms are fulfilled. In Ref. 4, it is proved that these axioms are independent; moreover, the authors prove that the most commonly used conical approximations satisfy all the axioms considered in their general definition.

In the sequel, \( \overline{M} \) is the closure of \( M \), \( U(x) \) is a neighborhood of \( x \), and \( U_\alpha(x) \) is a neighborhood of \( x \) of radius \( \alpha \).

We consider the following conical approximations of the set \( M \) at a point \( x^0 \in X \):

(i) \[
Z(M, x^0) = \{x \in X \mid \exists \lambda > 0 \text{ such that, } \forall t \in (0, \lambda), x^0 + tx \in M\},
\]

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the cone of feasible directions to $M$ at $x^0$;

(ii) $F(M, x^0) = \{x \in X \mid \forall \lambda > 0, \exists t \in (0, \lambda) \text{ such that } x^0 + tx \in M\},$

the radial tangent cone to $M$ at $x^0$;

(iii) $D(M, x^0) = \{x \in X \mid \exists U(x), \exists \lambda > 0 \text{ such that, } \forall t \in (0, \lambda), \forall \bar{x} \in U(x), x^0 + t\bar{x} \in M\},$

the cone of interior displacements to $M$ at $x^0$;

(iv) $T(M, x^0) = \{x \in X \mid \forall U(x), \exists \lambda > 0 \text{ such that, } \forall t \in (0, \lambda), \exists \bar{x} \in U(x) \text{ such that } x^0 + t\bar{x} \in M\},$

the Bouligand tangent cone or contingent cone to $M$ at $x^0$;

(v) $E(M, x^0) = \{x \in X \mid \exists U(x), \exists \lambda > 0 \text{ such that, } \forall t \in (0, \lambda), \exists \bar{x} \in U(x) \text{ such that } x^0 + t\bar{x} \in M\},$

the cone of attainable directions to $M$ at $x^0$;

(vi) $Q(M, x^0) = \{x \in X \mid \exists U(x) \text{ such that, } \forall \lambda > 0, \exists t \in (0, \lambda), \exists \bar{x} \in U(x) \text{ such that } x^0 + t\bar{x} \in M\},$

the cone of quasi-interior directions to $M$ at $x^0$.

In general, these cones are not convex. In order to obtain some convexity properties of local approximations cones, Elster and Thierfelder modify the previous cones in such a way that the point $x^0$ can be varied too. So, e.g., for the cone $E(M, x^0)$, we obtain the following modified cone:

$$E_m(M, x^0) = \{v \in X \mid \forall U(v), \exists \lambda > 0, \exists U(x^0) \text{ such that, } \forall \bar{x} \in (U(x^0) \cap \mathcal{M}) \cup \{x^0\}, \forall t \in (0, \lambda), \exists \bar{v} \in U(v) \text{ such that } \bar{x} + t\bar{v} \in M\}.$$ 

Similarly, we can obtain the other modified cones $Z_m(M, x^0)$, $F_m(M, x^0)$, $D_m(M, x^0)$, $T_m(M, x^0)$, and $Q_m(M, x^0)$; see Refs. 3, 4.

It can be shown that the following inclusion relations hold:

$$D_m(M, x^0) \subseteq D(M, x^0) \subseteq Q(M, x^0) \subseteq Q_m(M, x^0),$$
$$Z_m(M, x^0) \subseteq Z(M, x^0) \subseteq F(M, x^0) \subseteq F_m(M, x^0),$$
$$E_m(M, x^0) \subseteq E(M, x^0) \subseteq T(M, x^0) \subseteq T_m(M, x^0).$$

Moreover, it can be stated that the cones in the first row are open and the cones in the third row are closed.