Two Types of Minimax Theorems for Vector-Valued Functions

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Abstract. It seems that minimax theorems for vector-valued functions found in recent papers have something in common. Taking note of this, we improve several results in the author's recent works and state two types of minimax theorems for vector-valued functions. One theorem refers to functions with some special convexity properties; the other theorem refers to separated functions of the type \( f(x, y) = u(x) + v(y) \). The proofs are based on the existence of weak cone saddle points of \( f \) and on a condition about a pointed convex cone which induces a partial ordering in the image space of \( f \). We need the condition \((C \setminus \{0\}) + \overline{cl} \, C \subseteq C\), which implies the Sterna-Karwat condition for a convex cone \( C \) of a Hausdorff topological vector space.

Key Words. Minimax theorems, vector-valued optimization, \( C \)-convex functions, properly quasi \( C \)-convex functions, weak cone saddle points, pointed convex cones.

1. Introduction and Preliminaries

Recently, some papers have appeared which are devoted to minimax theorems for vector-valued functions; see Refs. 1-5. Except for slight differences of terminology and setting, these papers give similar reasonable definitions for minimal and maximal points of a set in a vector space with respect to a partial ordering induced by a pointed convex cone. They investigate the relations between the sets

\[
\min_{x \in X} \max_{y \in Y} f(x, y) \quad \text{and} \quad \max_{y \in Y} \min_{x \in X} f(x, y).
\]

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In Refs. 1, 2, and 5, minimax results in finite-dimensional spaces are presented. In Refs. 3 and 4, minimax results in infinite-dimensional spaces are stated. In particular, Refs. 2-4 give interesting answers to some open questions raised in Ref. 1. In our opinion, the minimax theorems in the above papers have a common result: some minimax points dominate some maximin points with respect to the ordering \( \leq \); i.e.,

\[ z_1 \leq z_2, \]

for some

\[ z_1 \in \text{minimax } f(x, y) \quad \text{and} \quad z_2 \in \text{maximin } f(x, y). \]

The aim of this paper is to improve the author's minimax theorems of Refs. 2 and 3, and to give some sufficient conditions for the above state to hold. To this end, we adopt the same approach as in Refs. 2 and 3; that is, we employ several properties of cone extreme points.

The organization of the paper is as follows. In Section 2, we start with an observation that Condition 2.2 implies Condition 2.1 (Sterna-Karwat's condition). Based on this observation only, several results in Ref. 3 are improved in Sections 2 and 3 by simply applying similar arguments as in Ref. 3. Then, in Section 3, we state the two main theorems in this paper. They are two types of minimax theorems for vector-valued functions, which might give us some clues to answer the questions raised in Ref. 4. In particular, Theorem 3.1 generalizes the known results on sufficient conditions for a vector-valued function to hold the above state; that is, some minimax points dominate some maximin points. The theorem includes four conditions for a vector-valued function with some special convexity properties. We prove the two minimax theorems through the existence of weak cone saddle points of vector-valued functions.

There are several differences between the notations in Refs. 1-5. In order to avoid confusion, in this paper we try to use the most acceptable notations. Throughout the paper, let \( E, F, Z \) be three real Hausdorff locally convex topological vector spaces (l.c.s.). Let \( C \subset Z \) be a pointed convex cone [i.e., \( \alpha C \subset C \), for \( \alpha \geq 0 \); \( C \cap (-C) = \{0\} \); and \( C \) is convex], which induces a partial ordering \( \leq_c \) in \( Z \) as follows: for \( z_1, z_2 \in Z, \) we denote

\[ z_1 \leq_c z_2, \quad \text{whenever } z_2 - z_1 \in C. \]

Also, we assume that \( \text{int } C \neq \emptyset, \) where \( \text{int } C \) denotes the interior of the set \( C. \) Then, \( C^0 := (\text{int } C) \cup \{0\} \) is a nontrivial pointed convex cone and induces a partial ordering \( \leq_{c^0} \) weaker than \( \leq_c \) in \( Z. \) An element \( z_0 \) of a subset \( A \)