TECHNICAL NOTE

New Proof of a Theorem of F. Giannessi

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Abstract. The aim of this note is to give a new proof and some improvements of Theorem 2.1 of Ref. 1, which plays an important role in deriving various optimality conditions.

Key Words. Convex cones, tangent cone, conic extension, separation.

1. Introduction

We begin with some notations and definitions.

Let $X$ be a nonempty subset of the Euclidean space $\mathbb{R}^n$, let $F: X \rightarrow \mathbb{R}^k$ be a vector-valued function, and let $\mathcal{K}$ be a convex cone in $\mathbb{R}^k$ with vertex at the origin $0 \in \mathbb{R}^k$ and $0 \notin \mathcal{K}$. Consider the following set, which is called the conic extension of $F(X)$ with respect to the cone $-\text{cl} \mathcal{K}$,

$$
\mathcal{E} := F(X) - \text{cl} \mathcal{K},
$$

where $\text{cl} \mathcal{K}$ stands for the closure of $\mathcal{K}$.

In what follows, the symbol $(\cdot, \cdot)$ will denote the scalar product in a Euclidean space. The cone generated by a subset $M$ of $\mathbb{R}^k$ is defined as

$$
\text{cone} \ M := \{tx: x \in M, t \geq 0\}.
$$

A face of $\text{cl} \mathcal{K}$ is defined as the intersection between $\text{cl} \mathcal{K}$ and a supporting hyperplane for it.

By $T_x(h)$, we denote the Bouligand tangent cone to $\mathcal{E}$ at a point $h \in \text{cl} \mathcal{E}$, i.e.,

$$
T_x(h) := \{v \in \mathbb{R}^k: \exists t_n \downarrow 0 \text{ and } v_n \rightarrow v \text{ such that } h + t_nv_n \in \mathcal{E}, \text{ for each integer } n\}.
$$

Note that, at $h = 0$, this definition coincides with the definition of tangent cone adopted in Ref. 1, p. 336.

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2. Results

We now give a proof for Theorem 2.1 in Ref. 1, which is stated as follows.

**Theorem 2.1.** Suppose that the set $\mathcal{E} := F(X) - \text{cl } \mathcal{H}$ is convex and contains 0 in its closure, $\beta$ is a face of $\text{cl } \mathcal{H}$, and there exists no $x$ satisfying the condition

$$F(x) \in \mathcal{H} \quad \text{and} \quad x \in X.$$  \hspace{1cm} (1)

Then, $\beta$ is contained in every hyperplane which separates $\mathcal{E}$ and $\mathcal{H}$, if and only if

$$\beta \subseteq T_\mathcal{E}(0).$$  \hspace{1cm} (2)

**Proof.** Necessity. Suppose that $\beta$ is contained in every hyperplane which separates $\mathcal{E}$ and $\mathcal{H}$. We have to show that (2) holds. *Ad absurdum*, assume that $\beta \not\subseteq T_\mathcal{E}(0)$. Then, there exists $a_0 \in \beta$ satisfying $a_0 \not\in T_\mathcal{E}(0)$. Since $0 \in \text{cl } \mathcal{E}$ and $\mathcal{E}$ is a convex set, it follows that $T_\mathcal{E}(0)$ is a nonempty convex cone and $T_\mathcal{E}(0) = \text{cl}(\text{cone } \mathcal{E})$. The separation theorem (Ref. 2, Theorem 3.4 in Chapter 3) ensures the existence of a nonzero vector $\lambda \in \mathbb{R}^k$ and a number $\mu \in \mathbb{R}$ such that

$$\langle \lambda, v \rangle \leq \mu < \langle \lambda, a_0 \rangle, \quad \forall v \in T_\mathcal{E}(0).$$  \hspace{1cm} (3)

$0 \in T_\mathcal{E}(0)$ implies that $\mu \geq 0$, so that in (3) we may set $\mu = 0$,

$$\langle \lambda, v \rangle \leq 0 < \langle \lambda, a_0 \rangle, \quad \forall v \in T_\mathcal{E}(0).$$  \hspace{1cm} (4)

We claim that the hyperplane

$$S^0 := \{ y \in \mathbb{R}^k : \langle \lambda, y \rangle = 0 \}$$

separates $\mathcal{E}$ and $\mathcal{H}$.

On the one hand, as it has been noted,

$$T_\mathcal{E}(0) = \text{cl}(\text{cone } \mathcal{E});$$

then from (4), it follows that

$$\langle \lambda, v \rangle \leq 0, \quad \forall v \in \mathcal{E}. \hspace{1cm} (5)$$

On the other hand, we have

$$\langle \lambda, h \rangle \geq 0, \quad \forall h \in \mathcal{H}. \hspace{1cm} (6)$$

Indeed, if there exists $h \in \mathcal{H}$ such that $\langle \lambda, h \rangle < 0$, then taking an arbitrary point $x \in X$, we see that

$$\langle \lambda, F(x) - th \rangle \to +\infty, \quad \text{when } t \to +\infty.$$