Bounded State Problem for Hereditary Control Problems

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Abstract. In this paper, we characterize optimal pairs for a hereditary control problem where the state is constrained. We use relaxed controls and the technique of penalization.

Key Words. Hereditary control problems, relaxed controls, penalization techniques, Gateaux derivatives.

1. Introduction

A control process where the evolution of the state depends upon the prior history of the state and control and where the state is constrained is studied.

The problem where the state is not constrained was studied by Bates in Ref. 1, where he derives a set of optimality conditions. Later, Medhin studied the same problem in Ref. 2, where it is shown how to handle perturbation of the state for \(-r \leq t \leq 0\) and also obtain appropriate transversality result.

The work here is a continuation of the work in Ref. 2. In this paper, we deal with the bounded state problem. As in Ref. 2 we use penalization and obtain optimality conditions at an optimal pair.

2. Statement of the Problem

The problem is to minimize the cost

\[
\int_0^1 f^*(\lambda(\cdot), u(\cdot), t) \, dt,
\]

(1)

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under the conditions

\[ \dot{\phi}(t) = f(\phi(t), u(t), t), \quad \text{a.e. } t \in [0, 1], \]  
\[ \phi(t) = y(t), \quad -r \leq t \leq 0, \quad y \in C, \]  
\[ u(t) \in \Omega(t), \quad \text{a.e. } t \in [0, 1], \]  
\[ G(\phi(t)) \leq 0, \quad \text{a.e. } t \in [0, 1], \]  
\[ Q(\phi(0), \phi(1)) = 0, \]

where

\[ f(\phi(\cdot), u(\cdot), t) = \langle f^1(\phi(\cdot), u(\cdot), t), \ldots, f^n(\phi(\cdot), u(\cdot), t) \rangle, \]
\[ f'(\phi(\cdot), u(\cdot), t) = \langle f^0(\phi(\cdot), u(\cdot), t), f^1(\phi(\cdot), u(\cdot), t), \ldots, f^n(\phi(\cdot), u(\cdot), t) \rangle, \]
\[ f_i(\phi(\cdot), u(\cdot), t) = h_i(t, \phi(\cdot), u(t)) + \int_{-r}^t g_i(t, s, \phi(s), u(s)) \, ds, \quad 0 \leq i \leq n. \]

**Assumption 2.1.**

(i) Denote by I the unit interval \([0, 1]\) and the interval \([-r, t]\), \(r > 0\) by \(I'_{-r}\). Let \(X\) be an open interval of \(\mathbb{R}^n\) and \(U\) an open interval of \(\mathbb{R}^m\).

(ii) We denote by \(C(I'_{-r}, X)\) the space of continuous functions from \(I'_{-r}\) into \(X\) with the supremum norm. We denote by \(AC(I'_{-r}, X)\) the subspace of absolutely continuous functions from \(I'_{-r}\) into \(X\).

(iii) The functions \(g^1, \ldots, g^n\) from \(I \times I'_{-r} = X \times U\) into \(\mathbb{R}^n\) are continuous in all variables. Further, \(\partial_x g_i(t, s, x, u), 1 \leq i \leq n\), are continuous in all arguments. Here, \((t, s, x, u)\) is a generic point of \(I \times I'_{-r} \times X \times U\).

(iv) We assume that \(u(t) \in \Omega\), a.e., where \(\Omega\) is a fixed compact subset of \(U\).

(v) The functions \(h^i: C(I'_{-r}, X) \times U \to \mathbb{R}\) are measurable in \(t \in I\) and continuous in \(u \in U\). The functions \(h^i(t, \cdot, u)\) are Frechet differentiable as a map from \(C(I'_{-r}, X)\) into \(\mathbb{R}\). Further, the derivatives are continuous in the second and third arguments. We also assume that, if \(\phi_1, \phi_2 \in C(I'_{-r}, X)\) are such that \(\phi_1 = \phi_2\) on \(I'_{-r}\), then

\[ h(t, \phi_1(\cdot), u(t)) = h(t, \phi_2(\cdot), u(t)). \]