TECHNICAL NOTE

Games with Vector Payoffs

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Abstract. Two-person games are defined in which the payoffs are vectors. Necessary and sufficient conditions for optimal mixed strategies are developed, and examples are presented.

Key Words. Game theory, vector maximization, Pareto optimality, equilibrium points, minimax theorems.

1. Introduction

An unresolved question in game theory has been whether there exists a theory for vector payoffs similar to standard results. Blackwell established in Ref. 1 an asymptotic analog to the minimax theorem for such payoffs, however, that is used in repeated games. Recent work by Nieuwenhuis (Ref. 2) and Corley (Ref. 3) suggests a generalization to vector payoffs using the following notion of vector maximization (or efficiency or Pareto optimality). Let \( u = (u_1, \ldots, u_n), \ v = (v_1, \ldots, v_n) \in D \subseteq \mathbb{R}^n \). If \( u_i \leq v_i, \ i = 1, \ldots, n, \) and \( u_j < v_j \), for some \( j \), we write \( u < v \) or \( v > u \). The point \( u \in D \) is said to be a vector maximum of \( D \), denoted \( u \in \text{v max} \ D \), if \( u \leq v \), for all \( v \in D \). Vector minima and \( v \min \ D \) have similar definitions.

A two-person (noncooperative) bimatrix vector-valued game is defined as follows. Player I has \( r \) strategies and Player II \( s \) strategies. The payoff with respect to I is represented by an \( r \times s \) matrix \( A = [a_{ij}] \) of \( n \)-tuples \( a_{ij} = (a_{i1}, \ldots, a_{in}) \in \mathbb{R}^n \), so \( A \) comprises the \( n \) real payoff matrices \( A_k = [a_{ik}], \ k = 1, \ldots, n \). Similarly, the payoff with respect to II is represented by an \( r \times s \) matrix \( B = [b_{ij}] \) of \( n \)-tuples \( b_{ij} = (b_{ij}^1, \ldots, b_{ij}^n) \in \mathbb{R}^n \) determining \( n \) real payoff matrices \( B_k = [b_{ij}^k], \ k = 1, \ldots, n \). Thus, when I plays his \( i \)th strategy and II his \( j \)th strategy, the payoff is \( (a_{ij1}, \ldots, a_{ijn}) \) to I and \( (b_{ij1}, \ldots, b_{ijn}) \) to II. Mixed strategies are allowed as usual. Player I assigns a probability \( x_i \).
to choosing strategy \( i \) and II assigns \( y_j \) to strategy \( j \). These mixed strategies are given by the column vectors by the column vectors \( x = (x_1, \ldots, x_r)' \) and \( y = (y_1, \ldots, y_s)' \) that are members, respectively, of the sets

\[
X = \left\{ x: \sum_{i=1}^{r} x_i = 1, x_i \geq 0, i = 1, \ldots, r \right\},
\]

\[
Y = \left\{ y: \sum_{j=1}^{s} y_j = 1, y_j \geq 0, j = 1, \ldots, s \right\}.
\]

The expected payoff of this game is therefore

\[
x'Ay = \sum_{i=1}^{r} \sum_{j=1}^{s} (x_iy_ja_{ij}^1, \ldots, x_iy_ja_{ij}^n) = (x'A_1y, \ldots, x'A_ny), \quad (1)
\]

for I, and

\[
x'By = \sum_{i=1}^{r} \sum_{j=1}^{s} (x_iy_jb_{ij}^1, \ldots, x_iy_jb_{ij}^n) = (x'B_1y, \ldots, x'B_ny), \quad (2)
\]

for II. The strategy pair \((x^*, y^*)\) is said to be an equilibrium point for this game if

\[
\hat{x}'A\hat{y} \not\leq x'A\hat{y}, \quad x \in X, \quad (3)
\]

and

\[
\hat{x}'B\hat{y} \not\leq \hat{x}'B\hat{y}, \quad y \in Y; \quad (4)
\]

the associated expectations \(\hat{x}'A\hat{y}\) and \(\hat{x}'B\hat{y}\) are called equilibrium values. When \(B = -A\), the result is a zero-sum vector-valued game. In that case (3), (4) can be combined into

\[
x'A\hat{y} \not\leq \hat{x}'A\hat{y} \not\leq \hat{x}'A\hat{y}, \quad x \in X, y \in Y, \quad (5)
\]

and \(\hat{x}'A\hat{y}\) is the equilibrium value. It is obvious that the above definitions reduce to the standard game-theoretic concepts (see Refs. 4 and 5) when \(n = 1\).

Two related notions for zero-sum vector-valued games are generalized minimax and maximin points. A strategy pair \((\hat{x}, \hat{y})\) for a zero-sum vector-valued game is said to be a minimax point if \(\hat{x}'A\hat{y}\) is a member of

\[
v \min \left[ \bigcup_{y \in Y} v \max \{x'Ay: x \in X\} \right]; \quad (6)
\]

\(\hat{x}'A\hat{y}\) is the associated minimax value. A maximin point is similarly defined with respect to the set

\[
v \max \left[ \bigcup_{x \in X} v \min \{x'Ay: y \in Y\} \right]. \quad (7)
\]