Stokes flow past a porous sphere using Brinkman’s model

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1. Introduction

In the calculation of Stokes flow past a porous particle, there have been many attempts to find the form of the equations of motion which model the flow in the porous medium accurately [1–3]. Experiments have been carried out by Beavers and Joseph [1] who suggested employing Darcy’s law in the interior of the body and allowing a “slip condition” at the surface of the body. To model the flow in some porous materials, a modification of Darcy’s law was proposed by Brinkman [2] and Debye and Bueche [4] independently. The validity of this equation was confirmed by the experimental verification of Matsumoto and Suganuma [3] for some materials. This model was used by Felderhof and Deutch [5a,b] and Felderhof [6a,b] to study the interactions between polymer molecules.

The problem of Stokes flow past porous particles using Brinkman’s equations in the interior was studied by Higdon and Kojima [7]. They derived some asymptotic results for small and large permeabilities by using Green’s functions and constructing integral solutions in the exterior and interior regions. Recently, a cartesian tensor solution of Brinkman’s equations was given by Yu and Kaloni [8]; the hydrodynamic force on a porous sphere in a uniform flow was calculated.

In this paper, we consider a general non-axisymmetric Stokes flow past a stationary porous sphere (using Brinkman’s model) in a viscous, incompressible fluid. We propose a representation of the velocity and the pressure fields for the Brinkman’s equations similar to the one suggested by Palaniappan et al. [9] for Stokes flow. This representation which uses two scalar quantities \( A' \) and \( B' \) (see Section 2) has the following advantages: For any given basic flow, \( A' \) and \( B' \) can be computed easily and so the exact solution for any given non-axisymmetric flow can be obtained. It also enables us to derive Faxén’s law [10] for drag and torque for a porous sphere. Some illustrative examples are worked out.

2. Method of solution

Consider a non-axisymmetric Stokes flow past a stationary porous sphere of radius \( 'a' \) in a viscous, incompressible fluid. For the region \( r < a \), we consider the Brinkman’s model. The equations of motion are

\[
\nabla p' + \mu \nabla^2 V' = \frac{\mu}{k} V' \\
\n\nabla \cdot V' = 0
\]
where $\mu$ is the co-efficient of dynamic viscosity, $k > 0$ is the permeability co-efficient of the sphere and the superscript \('i'\) is used to indicate the inner flow quantities. We assume a representation for $V^i$ and $p^i$ as follows

$$
V^i = \text{curl curl}(A^i) + \text{curl}(B^i) \tag{3}
$$

$$
p^i = p_0 + \mu \frac{\partial}{\partial r} [r(\nabla^2 - \lambda^2)A^i], \quad \lambda^2 = \frac{1}{k} \tag{4}
$$

where $p_0$ is a constant and

$$
\nabla^2(\nabla^2 - \lambda^2)A^i = 0, \quad (\nabla^2 - \lambda^2)B^i = 0. \tag{5a,b}
$$

Without loss of generality, we can write $A^i = A_1^i + A_2^i$, where $A_1^i$ and $A_2^i$ are respectively the solutions of

$$
\nabla^2 A_1^i = 0, \quad (\nabla^2 - \lambda^2)A_2^i = 0. \tag{6a,b}
$$

The flow in the region $r > a$ is governed by the Stokes equations

$$
\mu \nabla^2 V^e = \nabla p^e \tag{7}
$$

$$
\nabla \cdot V^e = 0. \tag{8}
$$

(We use the superscript \('e'\) for outer flow quantities.)

Equations (7)-(8) admit a representation for velocity and pressure as follows [9]

$$
V^e = \text{curl curl}(A^e) + \text{curl}(B^e) \tag{9}
$$

$$
p^e = p_0 + \mu \frac{\partial}{\partial r} [r\nabla^2 A^e] \tag{10}
$$

$$
\nabla^2 A^e = 0, \quad \nabla^2 B^e = 0. \tag{11}
$$

The boundary conditions are [6a, 7, 8]

(1) continuity of velocity components on $r = a$,

(i) $q_r^e(a, \theta, \phi) = q_r^i(a, \theta, \phi)$ \tag{12}

(ii) $q_\theta^i(a, \theta, \phi) = q_\theta^i(a, \theta, \phi)$ \tag{13}

(iii) $q_\phi^i(a, \theta, \phi) = q_\phi^i(a, \theta, \phi)$ \tag{14}

(2) continuity of stress components on $r = a$

(i) $T_{rr}^e(a, \theta, \phi) = T_{rr}^i(a, \theta, \phi)$ \tag{15}

(ii) $T_{r\theta}^e(a, \theta, \phi) = T_{r\theta}^i(a, \theta, \phi)$ \tag{16}

(iii) $T_{r\phi}^e(a, \theta, \phi) = T_{r\phi}^i(a, \theta, \phi)$ \tag{17}

where $q_r^i, q_\theta^i, q_\phi^i$ and $q_r^e, q_\theta^e, q_\phi^e$ are the radial, transverse and azimuthal components of velocity outside and inside the sphere respectively, in spherical polar co-ordinates $(r, \theta, \phi)$. The components of stress are

$$
T_{rr}^e = -p^e + 2\mu \frac{\partial q_r^e}{\partial r} \tag{18}
$$

$$
T_{r\theta}^e = \mu \left[ \frac{1}{r} \frac{\partial q_\theta^e}{\partial \theta} - \frac{q_\theta^e}{r} + \frac{\partial q_\theta^e}{\partial r} \right] \tag{19}
$$

$$
T_{r\phi}^e = \mu \left[ \frac{1}{r \sin \theta} \frac{\partial q_\phi^e}{\partial \phi} - \frac{q_\phi^e}{r} + \frac{\partial q_\phi^e}{\partial r} \right] \tag{20}
$$

$T_{rr}^i, T_{r\theta}^i$ and $T_{r\phi}^i$ can be calculated in a similar manner.