The Generalized KP Hierarchy

C. S. XIONG

Department of Physics, University of Tokyo, Bunkyo-ku, Tokyo 113, Japan
e-mail: xiong@danjuro.phys.s.u-tokyo.ac.jp

(Received: 6 September 1994; revised version: 5 June 1995)

Abstract. The KP hierarchy has been extended to a large set of integrable hierarchies. The main idea is to utilize the additional symmetry of the KP hierarchy, although not commuting among themselves. But we found that their proper combinations do commute with the original KP flows and among themselves, so as to give rise to an enlarged integrable hierarchy, which we call the generalized KP hierarchy.


Key words: Kadomtsev–Petviashvili hierarchy, integrable hierarchies, commuting flows.

1. Introduction

The Kadomtsev–Petviashvili (KP) hierarchy means a set of infinitely many differential equations

$$\frac{\partial}{\partial t_r} L = [L^r_+, L], \quad \forall r \geq 1$$

with the Lax operator $L$ being a pseudodifferential operator of the first order

$$L = \partial + \sum_{i=1}^{\infty} u_i(x) \partial^{-i}.$$ 

As usual, $\partial = \partial/\partial x$, $\partial^{-1}$ is a formal integral operation over $x$, while the subscript ‘+’ means truncation to the nonnegative powers of $\partial$. The integrability of the KP hierarchy is guaranteed by the commutativity of the flows (11).

Another description of the KP hierarchy is zero-curvature representation

$$\frac{\partial}{\partial t_n} L_+^m - \frac{\partial}{\partial t_m} L_+^n = [L_+^n, L_+^m], \quad \forall n, m \geq 1.$$ (2)

In the particular case $n = 2, m = 3$, denoting $u = 2u_1$, we obtain

$$3 \frac{\partial^2 u}{\partial t_2^2} = \left( 4 \frac{\partial u}{\partial t_3} - uu'' - 6uu' \right)', \quad u' = \frac{\partial}{\partial x} u$$ (3)

This is the KP equation.
The Lax operator $L$ can be expressed in terms of the 'dressing' operator

$$L = K \partial K^{-1}, \quad K = 1 + \sum_{i=1}^{\infty} w_i \partial^{-i}$$

whose equations of motion are

$$\frac{\partial}{\partial t_r} K = -L^r K, \quad \forall r \geq 1. \quad (5)$$

Another useful pseudodifferential operator is

$$M \equiv K \left( \sum_{r=1}^{\infty} r \partial_r \partial^{-1} \right) K^{-1} = \sum_{i=-\infty}^{\infty} v_i \partial^i,$$

which is conjugate to the KP operator $L$ in the sense that

$$[L, M] = K \left[ \partial, \sum_{i=1}^{\infty} r \partial_r \partial^{-1} \right] K^{-1} = 1. \quad (7)$$

The equations of motion for $M$ can be easily derived from Equations (5) and (6)

$$\frac{\partial}{\partial t_r} M = [L^-_r, M], \quad \forall r \geq 1. \quad (8)$$

The interesting question is whether we can introduce some other flows for this KP system. The answer is 'yes', the new series of flows can be defined as follows

$$\frac{\partial}{\partial t_{m,n}} L = [L, (L^m M^n)_-], \quad \forall m, n \geq 0. \quad (9)$$

The subset $(n = 0, m \geq 1)$ recovers the original KP flows (1), while the others are referred to as the additional symmetry of the KP hierarchy. The reason for giving such a name is due to the fact that each of these new flows commutes with the original KP flows (1), but they do not commute among themselves (for this part of the standard context, see reviews [1, 2]).

In this short Letter, we are going to show that there exist a set of new differential operators, which generate commuting flows on the phase space of the KP hierarchy. Each of these differential operators is a proper combination of the generators of the additional flows, i.e. it has the form

$$\sum_{a,b} f_{ab} L^a M^b$$

with $f_{ab}$ as functions on the parameter space. In other words, utilizing the additional symmetry of the KP hierarchy, we get a set of enlarged integrable hierarchies which we call the generalized KP hierarchy (see Equations (20a)-(20d) below).