LITERATURE CITED


SPACE AND FIELD GEOMETRY

Z. V. Khukhunashvili

In [1] we constructed a theory of geometric objects in which a transformation of the x-space arbitrarily depends on the geometric objects. In the current work we will continue the development of the mathematical apparatus to clarify the links between field theory and the geometry of 4-space. Concretely, we will study the geometry of the space $\Gamma_0^N \times J^R_N$ (topological product of the x-space and the inner space $J^R_N$ [1]).

1. Parallel Displacement

Suppose we have the transformation of the space $\Gamma_0^N \times J^R_N$ [1]

$$
\bar{x}^i = x^i + \lambda^i(x, w), \quad \bar{w}^k = w^k + \Omega^k(x, w),
$$

(1.1)

where $\Omega^k$ is the operator (4.3) [1]. The transformation (1.1) will be called the fundamental group of the space $\Gamma_0^N \times J^R_N$. For the sake of brevity of notation we will henceforth use the notation

$$
H^k = \Omega^k(x, w).
$$

(1.2)

We assume that in a neighborhood of the point $M_0 \in \Gamma_0^N$ there exists a field of the geometric object $w^K = \psi^K(x)$. Then the components $P^i = \frac{d\psi^K}{dx^i}$ are transformed by the law [1]

$$
P'^i = P^i + \frac{\partial H^k}{\partial x^i} + \frac{\partial H^k}{\partial w^k} P^i - \frac{\partial \lambda^i}{\partial x^i} P^i - \frac{\partial \lambda^i}{\partial w^k} P^k P^i.
$$

(1.3)

Let us consider the object $\psi^K$ at the adjacent point $M_1 (x + dx)$. Clearly, $\psi^K_{M_1} = \psi^K (x + dx)$. We displace $\psi_{M_1}$ parallel from the point $M_1$ to the point $M_0$ along some smooth curve connecting $M_1$ and $M_2$. The increment in the geometric object $\psi$ as it moves from $M_1$ to $M_0$ by parallel displacement will be the sum of the following increments:

1. Since $\Gamma_0^N$ is curved, in a parallel displacement of $\psi$ from $M_1$ to $M_0$ we must take into account the different orientations of the x-systems at these points. Therefore, in translating $\psi_{M_1}$ to $M_0$, the x-system must be contracted in such a way that $\psi_{M_1}$ and $\psi_{M_0}$ can be compared within a single x-system. Consequently, an increment in $\psi$ proportional to $dx^\nu$ occurs.

2. The variation due to the fact that $\psi$ is the field of a geometric object, i.e., $w^K = \psi^K(x)$. In this case we will have $d\psi^K = P^K dx^\nu$.

3. The increment $d\psi^K$ induces variations in the w-system of the curved space $J^R_N$. Therefore, the w-system must be rotated if $\psi_{M_1}$ and $\psi_{M_0}$ are to be in the same w-system. This operation implies an increment in the $\psi$-field proportional to the increment $d\psi$.

Thus, if $\tilde{\psi}_{M_1}$ is a geometric object that is moved by parallel displacement from $M_1$ to $M_0$,

$$
\tilde{\psi}_{M_0} = \tilde{\psi}_{M_1} + A^K dx^i + d\psi^K + A^K d\psi^i
$$

(1.4)

where $A^K_i$ and $A^K_i$ are functions of $x$ and $w$. We set

$D^x = d\omega^x + A_\nu^x dx^\nu + A_\nu^x dw^\nu. \quad (1.5)$

$D^K$ will be called the absolute differential of the geometric object $w^K$.

On the other hand, it follows from the fundamental group (1.1) that the space $\Gamma^N_0 \times J^N_0$ is curved, that is a variation in $\psi^K$ induces a variation in the reference grid of the x-space. The increment in $\psi$ in the course of displacing $\psi_{M_1}$ from $M_4$ to $M_5$ will therefore induce a displacement of the x-system from $M_0$ to a second point $M_2$. Thus the x-system must undergo a parallel displacement from $M_2$ to $M_0$, and this displacement will be proportional to $dx^\nu$ and $dw^K$:

$$\tilde{x}_{M_0} = x^\nu + A_\nu^x dx^\nu + A_\nu^x d\psi^\nu.$$  

The expression

$$D^* = dx^\nu + A_\nu^x dx^\nu + A_\nu^x dw^\nu, \quad (1.6)$$

where $A_\nu^\nu$ and $A_\nu^K$ are functions of x and w, will be called the absolute differential of $x^\nu$.

We introduce the matrix

$$\epsilon = \begin{pmatrix} e_\nu^x \\ e_\nu^K \end{pmatrix} \quad (1.7)$$

where

$$e_\nu^x = \delta_\nu^x + A_\nu^x, \quad e_\nu^K = A_\nu^K, \quad e_\nu^x = \delta_\nu^x + A_\nu^{-}. \quad (1.8)$$

Clearly, (1.7) carries $(dx^\nu, dw^K)$ into $(D^\nu, D^K)$. Let us assume that

$$\det \epsilon \neq 0.$$

The absolute differential (1.5) can be represented, by means of Eq. (1.6), in the form

$$D^* = Q^x D^x, \quad (1.9)$$

where

$$Q^x = (P_\nu^x + A_\nu^x + A_\nu^K P^K_\nu) R^x, \quad (1.10)$$

and $R^x$ is the inverse matrix to $\delta_\nu^\nu + A_\nu^\nu + A_\nu^K P^K_\nu$.

The set of quantities (1.10) will be called the covariant derivative of the field $\psi^K$.

2. The Transformations $D^\nu$ and $D^K$

It is known from the classical theory that the laws for transforming the cotangent vectors $dx^\nu$ and $dw^K$ have the form $dx^\nu = dx^\nu + (\partial \lambda^\nu / \partial x^\nu) dx^\sigma$ and $dw^K = dw^K + (\partial H^K / \partial w^K) dw^\nu$ when $x^\nu = x^\nu(x)$ and $w^K = w^K(w)$. These transformational properties of the differentials must be preserved if these vectors are to remain cotangent vectors when $H^K$ and $\lambda^\nu$ are localized with respect to x and w, respectively. But localization leads to the nonhomogeneous terms $(\partial \lambda^\nu / \partial w^K) dw^\nu$ and $(\partial H^K / \partial x^\nu) dx^\nu$ in the transformations $dx^\nu$ and $dw^K$. To eliminate them, it is necessary to introduce compound differentials. Based on the geometric structure of $D^\nu$ and $D^K$ they must clearly be identical with these new compound differentials. Thus, it is necessary that

$$\tilde{x}^\nu = x^\nu + A_\nu^x dx^\nu + A_\nu^K dw^K \quad (2.1)$$

where $H^K$ is (1.2). Equations (1.9) and (2.1) immediately imply that

$$Q^\nu = Q^x - \frac{\partial \lambda}{\partial x^\nu} Q^x + \frac{\partial H}{\partial w^K} Q^x. \quad (2.2)$$

Equation (1.6) is substituted in Eq. (2.1) in order to find the transformational properties of the coupling coefficients $A_\nu^\nu$ and $A_\nu^K$:

$$d\bar{x}^\nu + C^\nu d\bar{x}^\nu = dx^\nu + C^\nu dx^\nu + \frac{\partial \lambda^\nu}{\partial x^\nu} (dx^\nu + C^\nu dx^\nu), \quad (2.3)$$

where $C_\nu^K = A_\nu^\nu + A_\nu^K P^K_\nu$. Since

$$d\bar{x}^\nu = dx^\nu + \frac{\partial \lambda^\nu}{\partial x^\nu} dx^\nu + \frac{\partial \lambda^\nu}{\partial w^K} P^K_\nu dx^\nu, \quad (2.4)$$