Monodromy in the champagne bottle*

By Larry M. Bates, Dept of Mathematics, University of Calgary, Calgary, Alberta, Canada T2N 1N4

1. Introduction

Some years ago potentials of the form of an $S^1$ symmetric double well were of interest to field theorists studying the Higgs field. It turns out that such potentials are of interest even in classical mechanics, because they contain gross topological obstructions to the global construction of action-angle variables. To explain this, first recall that a Hamiltonian system on 2n-dimensional phase space is said to be completely integrable if one can find n functions in involution that are independent almost everywhere (usually we consider the Hamiltonian to be one of these n functions). In the usual case of interest in mechanics the common level set of the functions is a torus, and the motion on this torus is quasi-periodic. In more detail, one can find local canonical coordinates $(J_1, \ldots, J_n, \phi_1, \ldots, \phi_n)$ called action-angle variables on a neighbourhood of the torus so that the Hamiltonian depends only on the $J$s. Here the $\phi$s are angular coordinates on the torus. Such coordinate systems are desirable because one can read off the frequencies, the flow and other important dynamical information.

Note that the construction of action-angle variables is in general only a local one, and it is of some interest to see whether or not it can be extended to a global one. The obstructions to doing this are described in Duistermaat [6]. We will not treat the general problem, but restrict ourselves to the description of the particle in a champagne bottle potential. The potential is taken to be rotationally symmetric. Thus the Hamiltonian of classical form kinetic plus potential has the angular momentum as a conserved quantity. This gives two conserved quantities for a two degree of freedom system, and so the system is completely integrable. One then studies the image of the map $J$ from phase space $P$

$$J: P \rightarrow R^2$$

whose components are the angular momentum and the Hamiltonian. The

* Partially supported by NSERC grant OGP0042416.
regular values of this map are computed, and one can show that the inverse image of a regular value is connected and compact, and hence a two-torus. Letting \( R \) denote the set of regular values of \( J \), we now know that \( J^{-1}(R) \) has the structure of a torus bundle over \( R \). It turns out that \( R \) is not simply connected, and so the bundle need not be trivial. The monodromy of the bundle is then computed by using the system of parallel transport defined by the action variables. In a suitable basis it is found to be given by
\[
\begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix}.
\]

An exposition of monodromy stressing the role of the connection can be found in the appendix. We wish to emphasize that this is a known phenomena, having already been observed in the spherical pendulum [6] [4], the Lagrange top [5], and the Hamiltonian Hopf bifurcation [10]. The importance of the champagne bottle lies in its simplicity, as there is no complicated topology in the phase space, it being contractible.

2. The model

Define a potential in the plane by
\[
V(r) = r^4 - r^2,
\]
where \( r^2 = x^2 + y^2 \) and \( x \) and \( y \) are the usual Cartesian coordinates in \( \mathbb{R}^2 \). The Hamiltonian of a particle moving in the plane under the influence of this potential is
\[
h = \frac{1}{2}(p_x^2 + p_y^2) + (x^2 + y^2)^2 - x^2 - y^2
\]
in the usual canonical coordinates \((x, y, p_x, p_y)\). Changing to polar coordinates the Hamiltonian is
\[
h = \frac{1}{2}\left(p_\theta^2 + \frac{1}{r^2} p_\theta^2\right) + r^4 - r^2.
\]
Now
\[
\dot{p}_\theta = \{p_\theta, h\} = 0,
\]
since \( \theta \) is cyclic. Hence \( j = p_\theta \) is the conserved angular momentum. This means that the Hamiltonian system is completely integrable because we have the two conserved quantities \( j \) and \( h \) whose Poisson brackets vanish.