Smoothness and asymptotics of global positive branches of $\Delta u + \lambda f(u) = 0$

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Dedicated to Prof. Klaus Kirchgässner on the occasion of his sixtieth birthday

Introduction

The elliptic boundary value problem

$$\Delta u + \lambda f(u) = 0 \quad \text{in } \Omega \subset \mathbb{R}^2, \lambda \in \mathbb{R}, f(0) = 0,$$

$$u = 0 \quad \text{on } \partial \Omega,$$

was subject to innumerable investigations; we refer only to the reviews [1, 9] and to the monograph [15] where much more references can be found. Besides mere existence of a positive branch emanating at the smallest eigenvalue of $\Delta v + \lambda f'(0)v = 0$ the qualitative analysis of bifurcation diagrams in dependence of the growth of $f$ for large $u$ was of special interest.

The notion of a "branch", however, is somehow ambiguous; this vagueness is even increased when bifurcation diagrams are sketched: global branches are mostly presented as smooth curves, although the mathematical theory guarantees only continua. Since we do not know a proof in the literature we present here a fairly general result that positive branches are actually (unbounded) smooth curves.

When applied to the global branches with fixed nodal structure emanating at higher eigenvalues (see [7]), we get the result that these branches are also smooth curves in the appropriate fixed-point spaces with that precise nodal configuration. Thus secondary bifurcation from these curves are necessarily symmetry breaking bifurcations in this sense that the nodal lines are moved.

In the second part we study the asymptotic behavior of the positive smooth curve when $f$ has an exponential growth. The proofs go back to [13] where $f(u) = \sinh u$. Due to a slight generalization and due to the fact that this case is missing in the studies of bifurcation diagrams of [1, 9, 15], e.g., we think that it is useful to present it here.
Again this result applies to the unbounded curves with fixed nodal configuration emanating at higher eigenvalues: For each curve the same asymptotic behavior of the type \( \max |u| \leq C_1 - C_2 \ln \lambda \) holds true.

It is remarkable that problem (0.1) is not only a model for physics or biology (which are classical fields for possible applications) but that (0.1) with the particular nonlinearity \( f(u) = \sinh u \) over a rectangle \( \Omega \) arises also in differential geometry: The positive solutions of (0.1) for \( \lambda \) near zero (which exist by Theorem 2) are essential ingredients for the construction of compact surfaces of constant mean curvature. With these surfaces Wente [14] found counterexamples to a conjecture of H. Hopf (see also [13]).

1. Smoothness of the global positive branch

We assume that
\[
\Omega = \left(-\frac{a}{2}, \frac{a}{2}\right) \times \left(-\frac{b}{2}, \frac{b}{2}\right) \text{ is a rectangle in } \mathbb{R}^2, \ f: \mathbb{R} \to \mathbb{R}
\]
is of class \( C^2, f(0) = 0, f'(u) > 0 \) for \( u \geq 0 \), and \( f \) is odd.

The functional analytic setting of the boundary value problem (0.1) is the following: We define
\[
C_\lambda^{k,\alpha}(\mathbb{R}^2) = \{ u \in C^{k,\alpha}(\mathbb{R}^2), \ u \text{ is } 2a\text{-periodic in } x, \ u \text{ is } 2b\text{-periodic in } y \},
\]
where \( C^{k,\alpha}(\mathbb{R}^2) \) is the usual space of all \( k \)-times differentiable functions \( u \) such that \( u \) and its derivatives are (locally) Hölder continuous with exponent \( \alpha \in (0, 1) \). We use the Banach spaces
\[
D = C_\lambda^{2,\alpha} \cap \left\{ u \left( \frac{a}{2} + \cdot, y \right) \text{ and } u \left( x, \frac{b}{2} + \cdot \right) \text{ are odd} \right\}
\]
for all \((x, y) \in \mathbb{R}^2\), respectively
\[
E = C_\lambda^{0,\alpha} \cap \{ \cdot \}, \quad \text{endowed with the Hölder norms \( \| \cdot \|_{2,\alpha} \) and \( \| \cdot \|_{0,\alpha} \).}
\]
Then the left side of (0.1) defines a mapping
\[
G: \mathbb{R} \times D \to E, \ G(\lambda, u) = \Delta u + \lambda f(u), \text{ which is continuously Frechet differentiable;}
\]
\[
D_{(\lambda, u)}G(\lambda, u): \mathbb{R} \times D \to E, \text{ is given by}
\]
\[
D_{(\lambda, u)}G(\lambda, u)(\mu, v) = \mu f(u) + G_u(\lambda, u)v, \text{ where}
\]
\[
G_u(\lambda, u)v = \Delta v + \lambda f'(u)v.
\]