Couette flow between corrugated cylinders

By N. Phan-Thien, Dept of Mechanical Engineering, The University of Sydney, Sydney, N.S.W. 2006, Australia

1. Introduction

Stokes flows in corrugated channels and pipes have been considered by several authors, e.g., Chow and Soda [1], Wang [2], Phan-Thien [3]. In most cases, a sinusoidal corrugation is assumed, and a perturbation analysis in the amplitude of the corrugation is sought. Overall, it is difficult to obtain analytic solutions to the Stokes equations in a domain with a corrugated boundary. This Note reports some analytic Stokes solutions outside a rotating corrugated rod, and between two corrugated cylinders of specific cross sections. The corrugation is of a special form obtained from the epitrochoid and hypotrochoid conformal mappings [4]. There are two parameters in these mappings: one is the amplitude of the corrugation ($\varepsilon$) and the other ($n$) is related to the wavelength of the corrugation. These mappings have been used in a related Stokes flow problem in corrugated pipes [5]. The epitrochoid mapping has been used in the source-sink flow in a corrugated domain [6]. The problem may also be of some practical interest, since some commercial viscometers use corrugated cylinders to promote adhesion with the fluid being tested.

2. Analysis

We are concerned with the steady Stokes flow outside a rotating corrugated rod whose radius is given by $a = a(\theta)$, a continuous function of the polar angle $\theta$. The Stokes equations are

$$\eta \nabla^2 \mathbf{u} = \nabla P, \quad \nabla \cdot \mathbf{u} = 0,$$

where $\eta$ is the constant viscosity, $\mathbf{u}$ is the velocity vector, $\nabla$ is the gradient operator, and $P$ is the hydrostatic pressure. On the surface of the rod, $r = a(\theta)$, the fluid velocity is given by $\omega e_\theta$, where $\omega$ is the angular velocity of the rod, and $e_\theta$ is the unit vector in the $\theta$-direction. Far away from the rod, as $r \to \infty$, the velocity vector goes to zero. Later on, the flow between two corrugated cylinders is considered; one cylinder is rotating with an angular velocity $\omega$, and the other is held at rest (without loss of generality). Stick boundary conditions apply in this case also.

We can now normalize all distances by $L$, a length scale so that the radius of the rod varies between $1 \pm \varepsilon$, where $\varepsilon$ is the dimensionless amplitude of the corrugation. Velocities are normalized with respect to $\omega L$, so that the dimensionless angular velocity of the rod is unity. All stresses and pressure can be normalized with respect to $\eta \omega$. The flow is two-dimensional and continuity is satisfied by the introduction of the stream function $\chi$. 
where
\[ u_x = -\frac{\partial \chi}{\partial y}, \quad u_y = \frac{\partial \chi}{\partial x}, \]  
(1)

where \((u_x, u_y)\) are the Cartesian components of the velocity vector. The stream function satisfies the bi-harmonic equation:
\[ \nabla^4 \chi = 0 \quad \text{in } D, \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \nabla^4 = (\nabla^2)^2, \]  
(2)

where \(D\) is the region outside the rod (the flow domain). On the surface of the rod, where \(r = a(\theta)\), the velocity is purely rotational:
\[ \frac{\partial \chi}{\partial x} = x, \quad \frac{\partial \chi}{\partial y} = y. \]  
(3)

Far away from the rod, both these derivatives (velocity components) go to zero. The governing equations for the flow between two corrugated cylinders are the same, but the far-field boundary conditions now hold on the surface of the cylinder at rest.

2.1. Flow outside an hypotrochoid

A bi-harmonic function can be expressed in the form [4]
\[ \chi = \chi_0 + x\chi_1 - y\chi_2, \]  
(4)

where \(\chi_j, j = 0, 1, 2\) are harmonic functions, i.e., \(\nabla^2 \chi_j = 0\).

Laplace equations are invariant under conformal mappings, and thus \(\chi_j\) will also be harmonic in the region \(f(D)\), where \(f\) is any conformal mapping.

Instead of mapping a given region \(D\) with a corrugated boundary into a simple region, we adopt a particular conformal map and investigate the physical region that is mapped into the outside of a unit circle in the transformed plane. The particular map is the hypotrochoid [4]:
\[ z = x + iy = r e^{i\theta} = \zeta + \frac{\ell}{\zeta^{n-1}}, \]  
(5)

where \(n\) and \(\ell\) are constants, and the transformed domain \(\zeta\) is the outside of the unit circle in \(\zeta\)-plane:
\[ \zeta = \rho e^{i\phi}, \quad 1 \leq \rho < \infty, \quad -\pi \leq \phi \leq \pi. \]  
(6)

In terms of \(x\) and \(y\),
\[ x = \rho \cos \phi + \frac{\ell}{\rho^{n-1}} \cos(n-1)\phi, \]  
(7)
\[ y = \rho \sin \phi - \frac{\ell}{\rho^{n-1}} \sin(n-1)\phi, \]  
(8)

and in terms of the radius,
\[ r^2 = \rho^2 + \frac{\ell^2}{\rho^{2n-2}} + 2 \frac{\ell}{\rho^{n-2}} \cos n\phi. \]  
(9)

This map is conformal provided that \(|\ell|(n-1) < 1, \ n > 2\); it maps the outside of the unit circle \(1 \leq \rho < \infty\) in \(\zeta\)-plane into a compact region in \(z\)-plane with a corrugated boundary of \(n\) peaks. In fact, the radius of this boundary in \(z\)-plane (which corresponds