Complex variables and flexure

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Dedicated to Professor M. M. Abbassi

1. Introduction

In 1938, Stevenson [7] reduced the flexure problem of an isotropic beam to three Dirichlet boundary value problems (one of them is the torsion problem) together with three Neumann boundary value problems.

By using combinatorial identities Rung and Obaid [6] developed certain trigonometric identities which led to the solution of the flexure problem corresponding to the family of closed curves.

\[ r = a \cos^n \left( \frac{\theta}{n} \right), \quad a > 0, (-\pi < \theta \leq \pi), \quad n = 2, 3, 4, \ldots \quad (1.1) \]

These trigonometric identities are valid for \( n \geq k \geq 1 \), where both \( n, k \) are integers. The authors also conjectured the solution of the flexure problem corresponding to the closed curves \( C_n \)

\[ r = a \sin^{2n} \left( \frac{\theta}{2n} \right), \quad a > 0, (-\pi < \theta \leq \pi), \quad n = 1, 2, 3, 4, \ldots \quad (1.2) \]

In 1989, Obaid and Rung [4] used a different method which enabled the authors to generalize the original identities. The authors also introduced several new combinatorial, Chebyshev and Fibonacci identities.

The methods used earlier for solving the flexure problem corresponding to the curves (1.1) were complicated. In this paper we introduce new trigonometric and complex variable identities which lead to a straightforward solution of the flexure problem corresponding to the curves (1.2). A similar approach will simplify solving the flexure problem corresponding to the curves (1.1).

2. Fundamental equations

We assume that the Z-axis is taken parallel to the generators of the cylinder and the x-axis along the axis of symmetry of the cross section. The
cross section $S_n$ of the elastic isotropic cylinder is bounded by the curve $C_n$ in the $z$-plane $(z = x + iy)$. If the external forces are equivalent to the loads $(W_x, W_y, 0)$ acting at the centroid $x = h, y = 0$ of the end section, then Stevenson [7] reduced the problem to the determination of the following six analytic functions

$$w_{m,n} = \phi_{m,n} + i\psi_{m,n}, \quad \Omega_{m,n} = \Phi_{m,n} + i\Psi_{m,n} \quad (m = 1, 2, 3), \quad n = 1, 2, 3, \ldots$$

(2.1)

The harmonic functions $\psi_{m,n}, \Phi_{m,n}$ are determined by the boundary conditions

$$\psi_{m,n} = F_m, \quad \frac{\partial}{\partial y} (\Phi_{m,n} - F_m) = 0 \quad (m = 1, 2, 3), \quad n = 1, 2, 3, \ldots$$

(2.2)

where

$$F_1 = \frac{x^3}{3}, \quad F_2 = \frac{y^3}{3}, \quad F_3 = \frac{(x^2 + y^2)}{2},$$

(2.3)

and $y$ denotes the outward normal to the boundary $C_n$.

3. New trigonometric identities

The derivation of the six flexure functions for each curve of the family $C_n$ is dependent on the finding of three complex variable identities which can be deduced from four new trigonometric identities. We start by introducing the trigonometric identities and then we prove one of them. These identities are valid for all integers $n, k \geq 1$.

$$(-1)^{n-k}(2 \sin \phi)^{2n+2k} \cos(2n-2k)\phi$$

$$= \sum_{j=1}^{n} (-1)^{j}(\frac{2n + 2k - 2j - 1}{2k - 1})(2 \sin \phi)^{2j} \cos(2j\phi)$$

$$+ \sum_{j=1}^{k} (-1)^{j}(\frac{2n + 2k - 2j - 1}{2n - 1})(2 \sin \phi)^{2j} \cos(2j\phi)$$

$$+ \sum_{j=1}^{n} (-1)^{j-1}(\frac{2n + 2k - 2j}{2k - 1})(2 \sin \phi)^{2j-1} \sin(2j - 1)\phi$$

$$+ \sum_{j=1}^{k} (-1)^{j-1}(\frac{2n + 2k - 2j}{2n - 1})(2 \sin \phi)^{2j-1} \sin(2j - 1)\phi,$$

(3.1)