The acoustic impedance of a semi-infinite tube fitted with a conical flange

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We consider the propagation of acoustic waves in a semi-infinite tube, of constant cross section $\sigma$, whose centre-line lies along the negative $x$-axis. For $x > 0$ the tube is fitted with a conical flange, of solid angle $\omega$. In the following discussion, it is assumed that the tube and flange are of circular cross-section so that, strictly speaking, the results are valid only for this case. However, the final results are expressed in terms of $\sigma$ and $\omega$, since for only moderate departures from a circular cross-section it is likely that these are the operative parameters. Further comment on this generalisation will be made at a later stage.

We assume that the potential $\varphi_0$ describing the acoustic disturbance takes the form

$$\varphi_0 = e^{-ikx} - A e^{ikx}, \quad x \leq 0$$

$$= \frac{Be^{-ikr}}{r}, \quad r \geq r_0. \quad (1)$$

In the above expression for $x \leq 0$, the first term represents the incident wave, with wave-number $k$, and the second term represents the wave reflected from the open end $x = 0$. For $r \geq r_0$ the origin for $r$ is the apex of the conical flange, and $r_0$ is the value of $r$ at the lip of the open end (see Figure 1). Near $x = 0$ the detailed flow will be more complicated than that described by (1), but the expressions are still valid if we restrict attention to average values of $\varphi$, taken over the cross-section of the tube for $x \leq 0$ and over the cap of the sphere of radius $r$ bounded by the cone for $r \geq r_0$. Our purpose is to calculate the reflection and transmission coefficients $A$ and $B$. The discussion is a generalisation of that given by Rayleigh (1896, Volume 2, § 313) for a tube fitted with an infinite plane flange. It is restricted to long waves for which the flow between $x = 0$ and $r = r_0$ (the cap of the sphere bounded by the open end, see Figure 1) is essentially incompressible. One relation between $A$ and $B$ is then obtained from the condition that the flux across $x = 0$ is the same as the flux across $r = r_0$, or

$$- i\omega (1 + A) = - \omega B e^{-ikr_0} (1 + ikr_0). \quad (2)$$
A second condition is obtained from the fact that the difference in potential between 
\( x = 0 \) and \( r = r_0 \) is proportional to the flux. This gives

\[
\frac{B e^{-ikr_0}}{r_0} - (1 - A) = -ikc(1 + A),
\]

where \( c \) is the constant of proportionality. Elimination of \( B \) from (2) and (3) now gives

\[
\frac{1 - A}{1 + A} = \frac{ikc}{c + \frac{\sigma}{\omega r_0 (1 + ikr_0))}}.
\]

This is just the non-dimensional acoustic impedance of the tube, and the problem is now 
reduced to a calculation of \( c \). There are two existing calculations, both of which are better 
described with reference to an alternative approximation to (4). We write

\[
c + \frac{\sigma}{\omega r_0} = \left( \frac{\sigma}{\omega} \right)^{1/2} x
\]

so that \( x \) is now a non-dimensional function of \( \omega \). An approximate solution of (4) is then

\[
A = \exp \{-2ikx(\sigma/\omega)^{1/2} - 2k^2\sigma/\omega \}.
\]

This approximation is justified provided \( kr_0 \ll 1 \), which is a more restrictive inequality 
than that required for the validity of equations (2) and (3), at least for cones of small angle. 
In this form the second term of the exponent is described as radiation damping since it 
represents a loss of energy in the returning wave arising from radiation. The first term 
represents a phase shift. The length \( x(\sigma/\omega)^{1/2} \) is referred to as the end correction because, 
if the radiation damping is neglected, the potential can be calculated from the simple 
boundary condition of constant pressure, but applied at a distance \( x(\sigma/\omega)^{1/2} \) beyond the 
actual position of the mouth.

The first calculation of \( x \) was given by Rayleigh (1896, volume 2, appendix A) for a 
tube fitted with an infinite plane flange \( (\omega = 2\pi) \). The calculation depends on a minimum 
energy argument and gives

\[
x = 1.1656 \quad \text{for} \quad \omega = 2\pi.
\]

The second calculation for an unflanged tube was by Levine and Schwinger (1948), 
who give an exact solution of the acoustic problem based on a Wiener Hopf integral 
equation. Their result is

\[
x = 1.2266 \quad \text{for} \quad \omega = 4\pi.
\]

Here we contribute a third result for a cone of small angle. We first augment 
equation (1), in order to describe the flow near \( x = 0 \) in more detail, and write

\[
\begin{aligned}
\varphi &= e^{-ikx} - A e^{ikx} + \sum_n A_n J_0(\lambda_n \theta/\theta_0) e^{\lambda_n x/\theta_0}, \quad x \leq 0 \\
&= \frac{B e^{-ikr}}{r} + \sum_n B_n J_0(\lambda_n \theta/\theta_0)(r/r_0)^{-m}, \quad r \geq r_0
\end{aligned}
\]

where

\[
m(m - 1) = (\lambda_n/\theta_0)^2 = (\lambda_n r_0/\theta_0)^2
\]

approximately.

*) Note that \( r_0 = \left( \frac{\sigma}{\omega} \right)^{1/2} \left( 1 - \frac{\omega}{4\pi} \right)^{-1/2} \)