An integrable system of two nonlinear oscillators as attractor

By H. Tasso, Max-Planck-Institut für Plasmaphysik, Euratom Association, D-8046 Garching, W. Germany

In a previous paper [1] the Lyapunov stability and the location of attractors for a special system of nonlinear oscillators were investigated. This peculiar system is of the form

\[ \ddot{Y} + [(Y, A Y) M + (\dot{Y}, B \dot{Y}) N - P] \dot{Y} + CY = 0, \]  

(1)

where \( Y \) is a real vector of arbitrary length \( r \), \( A, B \) and \( C \) are \( r \times r \) positive definite symmetric matrices, and \( M, N \) and \( P \) are \( r \times r \) matrices whose symmetric part is positive definite. It is shown in Ref. [1] that the Lyapunov stability of system (1) can be discussed in terms of \( (Y, \dot{Y}) \) and \( (Y, C Y) \), and that the attractors have to be located in a strip of the first quadrant as drawn in Fig. 1.

\[ \beta_1, v_1, \pi_1 \] are the largest eigenvalues and \( \beta_0, v_0, \pi_0 \) the lowest eigenvalues of \( B, N \) and \( P \), respectively, \( N_0 \) and \( P \) being the symmetric parts of \( N \) and \( P \).

It is reasonable to expect limit cycles when the strip of Fig. 1 goes to zero. An example which was considered in Ref. [1] was to take

\[ A = \alpha I, \quad B = \beta I, \]
\[ M = \mu I, \quad N = v I, \]
\[ P = \pi I, \quad C = \frac{\alpha \mu}{\beta v} I = \epsilon I, \]

(2)

where \( I \) is the identity matrix. This leads [1] to a high-dimensional harmonic limit cycle. Notice that we can add to \( M, N \) and \( P \) antisymmetric parts \( M_a, N_a, P_a \) without altering the null thickness of the strip of Fig. 1. This brings the nonlinearities \( (Y, A Y) M_a, (\dot{Y}, B \dot{Y}) N_a \) into play and the problem of testing for a limit cycle becomes very difficult.
In this note the problem is restricted to an $r = 2$ system with the null thickness condition (see Ref. [1])

\[
(\dot{Y}, \dot{\dot{Y}}) + \epsilon (Y, Y) = \frac{\pi}{\beta \nu} \tag{3}
\]

\[
M_a = \begin{pmatrix} 0 & m/z \\ -m/z & 0 \end{pmatrix}, \quad N_a = \begin{pmatrix} 0 & n/\beta \\ -n/\beta & 0 \end{pmatrix}, \quad P_a = \begin{pmatrix} 0 & -p \\ p & 0 \end{pmatrix}. \tag{4}
\]

The null thickness condition (3) corresponds to the balance between the driving linear term $P_a \dot{Y}$ and the damping nonlinear terms $[(Y, A Y) M_a + (\dot{Y}, B \dot{Y}) N_a] \dot{Y}$.

This means that after the system has reached asymptotically the strip of zero thickness the equations of motion are given by

\[
\begin{align*}
\ddot{y}_1 + [m(y_1^2 + y_2^2) + n(y_1^2 + y_2^2) + p] \dot{y}_2 + \epsilon y_1 &= 0, \\
\ddot{y}_2 - [m(y_1^2 + y_2^2) + n(y_1^2 + y_2^2) + p] \dot{y}_1 + \epsilon y_2 &= 0. \tag{5}
\end{align*}
\]

System (5) is conservative and condition (3) expresses its energy integral. The nonlinear terms in system (5) correspond to some gyroscopic forces.

If system (5) is integrable, this means that system (1) under conditions (3) and (4) has a stable four-dimensional attractor. We want to prove integrability. Let us set

\[
z = y_1 + i y_2 \quad \text{and} \quad \bar{z} = y_1 - i y_2. \tag{6}
\]

System (5) can then be written as

\[
\ddot{z} - i [m z \ddot{z} + n \dot{z} \ddot{z} + p] \dot{z} + \epsilon z = 0. \tag{7}
\]

The polar representation of $z$ is

\[
z = \rho(t) e^{i \theta(t)}. \tag{8}
\]

Expression (8) is substituted in Eq. (7) and real and imaginary parts are separated to yield

\[
\begin{align*}
\dot{\rho} - \rho \dot{\theta}^2 + m \rho^3 \dot{\theta} + n \rho \dot{\theta} (\dot{\varphi} + \rho^2 \dot{\theta}^2) + \rho \pi \dot{\theta} + \epsilon \rho &= 0, \\
\rho \dot{\theta} + 2 \dot{\theta} \dot{\varphi} - m \rho^2 \dot{\theta} - n \dot{\varphi} (\dot{\varphi}^2 + \rho^2 \dot{\theta}^2) - p \dot{\varphi} &= 0. \tag{9}
\end{align*}
\]

Multiply Eq. (9) by $\rho$ and Eq. (10) by $\rho \dot{\varphi}$ and add

\[
\dot{\rho} \dot{\varphi} + \dot{\theta} \rho \dot{\theta} + \rho^2 \dot{\theta} \dot{\varphi} + \epsilon \rho \dot{\varphi} = 0. \tag{11}
\]

Equation (11) can be integrated once to

\[
\dot{\rho}^2 + \dot{\theta}^2 \rho^2 + \epsilon \rho^2 = h \tag{12}
\]

with $h = \pi/\beta \nu$ according to Eq. (3). The value of $\dot{\rho}^2 + \dot{\theta}^2 \rho^2$ from Eq. (12) is substituted in Eqs. (9) and (10) to yield

\[
\begin{align*}
\dot{\rho} - \rho \dot{\theta}^2 + m \rho^3 \dot{\theta} + n \rho \dot{\theta} (h - \epsilon \rho^2) + p \rho \dot{\theta} + \epsilon \rho &= 0, \\
\rho \dot{\theta} + 2 \dot{\theta} \dot{\varphi} - m \rho^2 \dot{\theta} - n \dot{\varphi} (h - \epsilon \rho^2) - p \dot{\varphi} &= 0. \tag{13}
\end{align*}
\]

Multiply Eq. (14) by $\rho$ and integrate once:

\[
\rho^2 \dot{\theta} = \frac{m}{4} \rho^4 + \frac{n h}{2} \rho^2 - \frac{\epsilon n}{4} \rho^4 + \frac{p}{2} \rho^2 + C, \tag{15}
\]