A note on gravity waves in a viscous liquid with surface tension

By Peder A. Tyvand, Dept. of Mechanics, University of Oslo, Norway

Reid [1] investigated the gravitational stability of two superposed semi-infinite viscous fluids with interfacial tension $T$. When one fluid has vanishing density, the characteristic equation of linear theory reduces to:

$$y^4 + 2y^2 - 4y + 1 \pm \hat{k}^{-3} + S\hat{k}^{-1} = 0. \quad (1)$$

Here $y$ is given by

$$y = \left(1 + \frac{\sigma}{k^2 v}\right)^{1/2} \quad (2)$$

where $\sigma(= \sigma_e + i\sigma_i)$ is the amplification factor, $k$ the wave number and $v$ the kinematic viscosity.

In eq. (1) the plus signs refers to the case where the upper fluid has zero density (static stability), and the minus sign refers to the case where the lower fluid has zero density (static instability). $\hat{k}$ and $S$ denote dimensionless wave number and surface tension, respectively. Following Reid [1] we also define a dimensionless amplification factor $\hat{\sigma}$:

$$(\hat{k}, \hat{\sigma}, S) = \left(\frac{y^{2/3}}{g^{1/5}} k, \frac{v^{1/3}}{g^{2/3}} \sigma, \frac{T}{g^{1/3} \rho v^{4/3}}\right) \quad (3)$$

Reid’s surface tension parameter $S$ may be expressed by

$$S = M^{-1/3} \quad (4)$$

where $M$ is the Morton number

$$M = \frac{g \mu^3 v^4}{T^3} = \frac{g \rho^4}{\rho T^3} \quad (5)$$

Here $\mu(= \rho v)$ is the dynamic viscosity of the liquid. The Morton number is important in connection with motion of bubbles through liquids, see e.g. Harper [2].

Lighthill [3, p. 223] has pointed out that surface tension, in linear theory, only modifies the effect of gravity. Introduction of surface tension may be expressed by the transformation

$$g \rightarrow g \pm \frac{1}{\rho} T k^2 \quad (6)$$

where the plus sign refers to static stability and the minus sign to static instability. For the latter case, zero net effect of gravity occurs for the neutral wave number given by;

$$k_n = \left(\frac{\rho g}{T}\right)^{1/2} \quad (7)$$
or equivalently, $\hat{k} = S^{1/2}$. The principle expressed by the transformation (6) may be applied directly to our problem by the transformed quantities defined by:

$$\hat{k} = (\pm \hat{k}^{-3} + S \hat{k}^{-1})^{-1/3}$$

$$\hat{\sigma} = \frac{\hat{k}^2}{k^2} \hat{\sigma}.$$  \hfill (9)

In eq. (8), as in eqs. (1) and (6) above, the plus sign refers to static stability and the minus sign to static instability.

The transformed dispersion relation $\hat{\sigma} = \hat{\sigma} (\hat{k})$ is independent of surface tension, and was computed by Chandrasekhar [4], who worked out the theory given by Lamb [5, p. 625]. We will here only list some asymptotic expressions for $\hat{\sigma} (\hat{k})$:

We have Rayleigh-Taylor instability when $\hat{k} < 0$. We find the following two-term asymptotic expansions:

$$\hat{\sigma} \sim \frac{1}{\hat{k}} - 2 \hat{k}^2; \quad |\hat{k}| \ll 1 \quad (10)$$

$$\hat{\sigma} \sim \frac{1}{2|\hat{k}|} - \frac{3}{16|\hat{k}|^4}; \quad |\hat{k}| \gg 1. \quad (11)$$

Eq. (10) represents the inviscid solution plus its first correction, while eq. (11) represents the creeping motion solution [6] plus its first correction.

When $\hat{k} > 0$ we have the surface wave problem. For $\hat{k} < \hat{k}_b = 1.1981$ [4] the solution is periodic in time, and an asymptotic formula for the dispersion relation is

$$\hat{\sigma} \sim 2 \hat{k}^2 + i(\hat{k}^{1/2} - \sqrt{2}\hat{k}^{1/4}); \quad \hat{k} \ll 1 \quad (12)$$

giving the imaginary inviscid solution plus its first (complex) correction. For $\hat{k} > \hat{k}_b$ the solution is aperiodic in time and consists of two branches [4]: First the rapidly decaying "viscous" mode, for which we have the asymptotic formula

$$\hat{\sigma} \sim a \hat{k}^2 + b \hat{k}^{-1}; \quad \hat{k} \ll 1 \quad (13)$$

where $a$ and $b$ are determined by the equations

$$a^3 + 8a^2 + 24a + 16 = 0 \quad b = -\frac{a^2 + 4a + 4}{2a^3 + 12a^2 + 24a + 8} \quad (14)$$

giving $a = -0.91262197, b = 0.21780819$. The second (physically dominating) creeping mode satisfies the formula

$$\hat{\sigma} \sim -\frac{1}{2k} - \frac{3}{16k^4}; \quad k \gg 1 \quad (15)$$

asymptotically. By comparing with eq. (11), we note that the absolute rate of the first deviation from the creeping motion solution is the same as for Rayleigh-Taylor instability.

From the above asymptotic expressions for $\hat{\sigma}$ we may find corresponding expressions for $\sigma$ by utilizing (7)–(8). Only terms proportional to $k^2$ give contributions to $\sigma$ which are independent of surface tension.