EINSTEIN'S EQUATIONS AS EQUATIONS OF IMBEDDING
AND PRIVILEGED FRAMES

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It is shown that Einstein's equations for empty space in a synchronous frame are equations of imbedding of three-dimensional Riemannian spaces in four-dimensional pseudo-Riemannian spaces with $R_{ik} = 0$. Choice of the privileged synchronous frame enables one to distinguish the quantities which characterize the gravitational field from those that are due to the frame's being non-inertial. In particular, it is shown that the energy of the gravitational field does not contribute to the total matter energy for island systems, which agrees with Einstein's result for a spherical universe.

There are now no serious doubts about Einstein's equations, though their interpretations may differ. The general theory of relativity regards the gravitational field as a manifestation of the geometry of four-dimensional space—time. Further investigations have, however, shown that the dynamics of the gravitational field corresponds to the dynamics of a three-dimensional Riemannian space, with the geometrical structure of the theory remaining unaltered. In this paper it is shown that Einstein's equations for empty space in a synchronous frame are the equations of imbedding of three-dimensional Riemannian spaces in four-dimensional pseudo-Riemannian spaces with $R_{ik} = 0$, and it is therefore more natural to regard gravitational fields as a manifestation of the geometry of three-dimensional Riemannian spaces that vary in time and admit imbedding in four-dimensional Einstein spaces. Let us recall some of the results of imbedding theory.

Suppose a hypersurface $V_{n-1}$ in $V_n$ is specified by the equations

$$x^i = x^i(u^1, ..., u^{n-1}),$$

and the metric tensor of $V_{n-1}$ takes the form

$$g_{ij} = \frac{\partial x^i}{\partial u^a} \frac{\partial x^j}{\partial u^b} g_{ab},$$

(2)

The system of derivatives $\xi^{\alpha}_i = \partial x^i/\partial u^\alpha$ is called the mixed tensor of the hypersurface; the Latin index refers to the space $V_n$ and transforms as a contravariant index $\xi^{\alpha}_i = (\partial x^i/\partial u^\alpha)\xi^\alpha_i$, while the Greek index refers to $V_{n-1}$ and transforms as a covariant index $\xi^{\alpha}_{ij} = (\partial u^a/\partial u^\beta)\xi^\alpha_{ij}$. At every point of the hypersurface $V_{n-1}$ we construct a frame consisting of $n$ vectors $\xi_i, ..., \xi_{n-1}, \nu^i$, where $\xi_i, ..., \xi_{n-1}$ are linearly independent tangent vectors and $\nu^i$ is a unit (or imaginary unit) normal vector. The extrinsic geometry of the hypersurface is determined by the second fundamental tensor

$$b_{\alpha i} = \mp g_{ij} \nu^i \nu^j,$$

(3)

whose superscript corresponds to the unit vector and subscript to the imaginary unit vector $\nu^i$. The conditions of imbedding of $V_{n-1}$ in $V_n$ determine the Gauss—Peterson—Codazzi equations [1]:

$$R_{ik\lambda\nu} = R_{ik\rho\nu} \xi^\rho \xi_\lambda \xi^\nu \pm \left( b_{\lambda \nu} - b_{\rho \nu} - b_{\lambda \rho} \right),$$

$$- R_{ik\rho} \xi^\rho \nu^\nu = \mp \left( b_{\lambda \nu} - b_{\lambda \rho} \right).$$

(4)

It should be said that the treatment of the vectors $\xi^\alpha_i$ and $\nu^i$ as unknown quantities does not have a geometrical meaning since it is then necessary to assume that $u_1, ..., u_{n-1}, b_{\lambda \nu}$, and $\Gamma^\mu_{ik}$ are known functions, i.e., that one knows how $V_{n-1}$ is imbedded in $V_n$, $\xi^\alpha_i$ and $\nu^i$ then being known. An exception is only the case of flat space, when $R_{ik\lambda\nu} = 0$. Therefore, the problem is properly posed as follows: given the Riemann—Christoffel tensor...
of the imbedding space, given a coordinate system and the equation of the surface; then the first and the second fundamental tensors are determined uniquely. However, in the theory of gravitation the Riemann—Christoffel tensor $R_{ijkl}$ is usually unknown, but we shall show that if the Ricci tensor is equal to zero then in the semigeodesic coordinate system the Gauss—Peterson—Codazzi equations do not contain the components of the Riemann—Christoffel tensor. We write the equations of imbedding of $V_{n-1}$ in $V_n$ in the semigeodesic coordinate system ($\xi^1, \xi^2, \ldots, \xi^n$ is an imaginary unit vector):

$$^{3}R_{x_{1}x_{2}} = R_{x_{1}x_{2}} = (b_{x_{1}x_{2}} - b_{x_{2}x_{1}}),$$

$$-R_{x_{1}x_{2}} = (b_{x_{1}x_{2}} - b_{x_{2}x_{1}}),$$

where $^{3}R_{x_{1}x_{2}}$ is the Riemann—Christoffel tensor of the three-dimensional space. Since this tensor has only six linearly independent components, which is as many as in the Ricci tensor, as a system of linearly independent Gauss equations one can take the contracted equations (5). The number of linearly independent Peterson—Codazzi equations is three, and instead of them we can also write the contracted equations (6), and then the Gauss—Peterson—Codazzi equations take the form

$$^{3}R_{x_{3}x_{4}} = R_{x_{3}x_{4}} = (b_{x_{3}x_{4}} - b_{x_{4}x_{3}}),$$

$$-R_{x_{3}x_{4}} = (b_{x_{3}x_{4}} - b_{x_{4}x_{3}}).$$

The components $R_{x_{3}x_{4}x_{5}}$ can be expressed in terms of $b_{x_{3}x_{4}x_{5}}$

$$R_{x_{3}x_{4}x_{5}} = -\frac{\partial}{\partial x^{3}} b_{x_{4}x_{5}},$$

and if the Ricci tensor is zero in the imbedding space, the equations of imbedding are Einstein’s equations for empty space:

$$^{3}R_{x_{3}x_{4}} = \frac{\partial}{\partial x^{3}} b_{x_{4}} - (b_{x_{4}x_{4} - 2b_{x_{3}x_{4}}}),$$

$$b_{x_{3}x_{4}} = 0,$$

if one bears in mind that

$$b_{x_{4}} = -\sum_{i} \left( \Gamma_{ij}^{i} + \Gamma_{ij}^{i} \right) \delta_{j}^{i}.$$

reduces to the form

$$b_{x_{3}} = -\frac{1}{2} \frac{\partial b_{x_{4}}}{\partial x^{3}}.$$

A semigeodesic coordinate system is realized in the synchronous comoving frame in which point bodies move along geodesics without rotation or acceleration.

If it is assumed that the synchronous frame is privileged, then Einstein’s equations together with the hypersurface equation $x^{4} = \text{const}$ distinguish in this frame a three-dimensional Riemannian space which is defined by the initial data $g_{\alpha\beta}$ and $b_{\alpha\beta} = -(1/2)(\partial g_{\alpha\beta}/\partial x^{4})$. It is well known [2] that the system of equations (3) and (11) forms the Cauchy—Kovalevskaya problem, and specification of the initial data determines the development of the hypersurface in time.

It is therefore natural to assume that the three-dimensional Riemannian space imbedded in the Einstein space has a greater fundamental significance than the four-dimensional space—time, and one can regard gravitational fields as manifestation of the geometry of three-dimensional Riemannian spaces unfolding in time.

From this assumption there follows the requirement that the physical quantities characterizing the gravitational field should not depend on the choice of the spatial coordinate system in the synchronous frame. Indeed, the Christoffel symbols, which play the role of the field strengths in the Newtonian limit, have the following form in the synchronous frame:

$$\Gamma_{ij}^{i} = \Gamma_{ij}^{i} = 0, \Gamma_{ij}^{i} = -b_{ij}, \Gamma_{ij}^{i} = -b_{ij}, \Gamma_{ij}^{i} = \Gamma_{ij}^{i} = -b_{ij}^{i},$$

and decompose into the second fundamental tensor and the three-dimensional Christoffel tensors.

Of the energy—momentum pseudotensors, only that of Møller and Mitskevich [3] has energy and momentum components that do not depend on spatial transformations. Expressed in terms of the superpotential, this pseudotensor has the following form outside the sources: