Mather sets for plane Hamiltonian systems

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1. Introduction

Recently Mather (Ma) developed a theory for area preserving monotone twist maps of an annulus, the main result of which is the generalization of the notion of an invariant curve of such maps. An easy introduction to this theory can be found in (MK) or (Ba). See also (Mo 2). This theory can be applied to the Poincaré map of plane Hamiltonian systems, where the radial coordinate on the annulus corresponds to the momentum conjugated to the angle coordinate. The monotonicity condition for the map \((x, y) \mapsto (x_1, y_1)\) is:

\[
\frac{\partial x_1}{\partial y} > 0.
\]

For the time-\(\Delta t\)-map to satisfy the monotonicity condition the Legendre condition \(L_{x'x} \geq \delta > 0\) on the underlying Lagrangian is sufficient, provided \(\Delta t\) is sufficiently small, see Fig. 1.

But the smallness of \(\Delta t\) is not in general satisfied in applications. For example, in case of a periodic perturbation the Poincaré map is the period map and the period will not be small enough in general. Thus the theory of monotone
twist maps does not apply. But the ideas carry over to the more general case leading to a theory for plane periodic Hamiltonian differential equations. It is this generalization which is the goal of the present paper. Strictly speaking, it is not a full generalization, because Mather does not need differentiability assumptions. But given the necessary differentiability, Moser showed that an area preserving monotone twist map can be interpolated by a Hamiltonian system (Mo 1).

As pointed out above, the methods to be developed below are similar to those for monotone twist maps and explained in (Ba, MK). But since there is considerable interest in these topics presently, it seems appropriate to give a reasonably self-contained exposition of Mather’s theory.

For the relationship of Mather-Aubry theory to 1-dimensional crystals see (Au). The connections to geodesics on tori are described in (Ba), see also the literature cited there.

As a typical example for the continuous case the reader may wish to think of the mathematical pendulum with not necessarily small periodic perturbation. The corresponding Lagrangian is

$$L(t, x, x') = \frac{1}{2} x'^2 + V(t, x)$$  \hspace{1cm} (1.1)

where

$$V(t + 1, x) = V(t, x) = V(t, x + 1) \quad \text{and} \quad t, x, x' \in \mathbb{R}.$$  

The key idea in the sequence is to make use of the Lagrangian variational principle and to look for minimals rather than for extremals. Minimals in fact enjoy properties not shared by other solutions of Euler’s equations.

The theory developed below makes extensive use of the total ordering of the real line, thus it does not generalize to higher dimensional ODEs. For generalizations to PDEs, however, see (Mo 3).

2. Existence and properties of minimals

Let $L = L(t, x, x')$ be a $C^2(R^3)$ function 1-periodic both in $t$ and $x$,

$$\delta \leq L_{x', x'} \leq 1/\delta, \quad |L_x| \leq c \cdot (1 + x'^2), \quad |L_{x'}| + |L_{x', x}| \leq c \cdot (1 + |x'|). \quad (2.1)$$

Here and in the following the letter $c$ means some constant which may differ from equation to equation.

By integration we immediately get from this

$$-c + \delta |x'| \leq |L_x(t, x, x')| \leq c + |x'|/\delta$$

$$-c + \delta x'^2/4 \leq L(t, x, x') \leq c + x'^2/\delta$$

$$E(t, x, p^*, p) := L(t, x, p) - L(t, x, p^*) - (p - p^*) L_{x'}(t, x, p^*) \geq \delta (p - p^*)^2/2$$