A note on the propagation of circular fronts into still water

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1. Introduction

The one dimensional shallow water equations of Stoker [2] have been used extensively to study the propagation of plane disturbances up sloping beaches. Greenspan [3] obtained the solution for waves running up beaches of constant slope while Carrier and Greenspan [4] made an effort to discover a general criterion for wave breaking. The asymptotic theory of flow at the wavefront has been given by Bürger [5] to determine the point where the wave breaks. Jeffrey [6] established a general theory for the development and propagation of discontinuities in quasilinear hyperbolic systems of partial differential equations and successfully applied it to determine the position and the time at which a shock forms on the wavefront [7].

Recently, Gurtin [1] used a simple argument to find the condition for the breaking of finite amplitude water waves propagating into water at rest above a sloping beach of arbitrary shape. Jeffrey and Mvungi [8] generalized the method by Gurtin [1] to derive the amplitude of an acceleration wave propagating on the surface of water at rest in a vertical walled channel of arbitrary continuously varying width and depth. This same argument was extended by Jeffrey and Mvungi [9] to study the effect of submerged obstacles on water waves in a channel. It is the purpose of this paper to generalize the approach due to Gurtin [1] to obtain the explicit expression of the amplitude of circular fronts propagating into water at rest together with a criterion for the breaking or non breaking on such fronts.

2. Amplitude of circular fronts

We introduce cylindrical co-ordinates \( r, \phi \) and \( z \) such that the \( r, \phi \) axes lie on the equilibrium surface of the water with the \( z \)-axis pointing vertically upwards. Limiting the following to the propagation of dotting waves in shallow water, we start from the following quasilinear one dimensional equations

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + g \frac{\partial \eta}{\partial r} &= 0 \\
\frac{\partial \eta}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left[ (\eta + h) ru \right] &= 0
\end{align*}
\]

where \( u(r, t), \eta(r, t) \) and \( g \) are respectively, the \( r \)-component of water velocity, the free surface elevation and the acceleration due to gravity.
We suppose that circular fronts move in the direction of increasing $r$ and that the advancing front is at $r = r_0$ when $t = 0$. Like Gurtin [1] we assume that

(i) $u$ and $\eta$ are continuous, with $u(r, t) = \eta(r, t) = 0$ a head of the advancing wave. and (ii) the first and second derivatives of $u$ and $\eta$ suffer at most a jump discontinuity, so that the wave propagated on the surface is an acceleration front.

Using a superscript minus sign to denote the value of a function immediately behind the advancing wavefront we conclude from (i) that

$$u^- = \eta^- = 0. \quad (2.3)$$

Differentiation of (2.3) with respect to $t$ leads to

$$u_t^- = - \frac{du}{dt} \quad \text{and} \quad \eta_t^- = - \frac{d\eta}{dt}. \quad (2.4)$$

The characteristic slopes associated with (2.1) and (2.2) are

$$\frac{dr}{dt} = \lambda^{(1)} = u + \sqrt{g(\eta + h)} \quad \text{and} \quad \lambda^{(2)} = u - \sqrt{g(\eta + h)}. \quad (2.5)$$

These are real and distinct whence the quasilinear system (2.1), (2.2) is strictly hyperbolic. Now $\lambda^{(1)}$ represents the advancing wavefront whence (2.4) and (2.5) combine to yield

$$u_t^- = - u^- \sqrt{gh(r)} \quad \text{and} \quad \eta_t^- = - \eta^- \sqrt{gh(r)}. \quad (2.6)$$

If we define the amplitude of the acceleration wave to be

$$a = \eta_r^- \neq 0 \quad (2.7)$$

then (2.6) becomes

$$u_t^- = - \frac{u^-}{u} \sqrt{gh(r)} \quad \text{and} \quad \eta_t^- = - a \sqrt{gh(r)}. \quad (2.8)$$

Now, immediately behind the wavefront (2.1) and (2.2) become

$$u_t^- + g \eta_r^- = 0 \quad \text{and} \quad \eta_t^- + h u_r = 0. \quad (2.9)$$

Equations (2.8) and (2.9) combine to yield

$$u_t^- = - g a \quad \text{and} \quad u_r^- = \frac{g a}{\sqrt{gh}}. \quad (2.10)$$

Differentiating (2.1) with respect to $t$ and (2.2) with respect to $r$ and eliminating $\eta_{rr}$, we obtain, behind the wavefront,

$$u_t^- - g hu_{rr}^+ + u_t^- u_r^- - 2 g u_r^- \eta_r^- - 2 g h \eta_r^- u_r^- - g h \frac{u_r^-}{r} = 0. \quad (2.11)$$

Using (2.10) in (2.11) leads to

$$u_t^- - g hu_{rr}^- - \frac{g^2 a}{\sqrt{gh}} \left[ 3 a + 2 h + \frac{h}{r} \right] = 0. \quad (2.12)$$