On the Kolodner-Coffman method for the uniqueness problem of Emden-Fowler BVP*

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1. Introduction

The nonlinear second-order ordinary differential equation

\[ y''(t) + q(t)f(y(t)) = 0, \quad -\infty < a < t < b < \infty, \tag{1.1} \]

arises in many areas of applications. Here \( q: (a, b) \to \mathbb{R} \) and \( f: \mathbb{R} \to \mathbb{R} \) are continuous functions. It includes the important class of Emden-Fowler equations. Emden and Fowler studied equation (1.1) when \( q(t) \) is a power of \( t \) and \( f(t) \) is, roughly speaking, a power of \( y \). More precisely, they considered the particular form of (1.1)

\[ y'' + q(t)|y|^\gamma \text{sign } y = 0 \quad (0 < \gamma < \infty), \tag{1.2} \]

where \( q(t) = t^\alpha \) for some constant \( \alpha \). Portions of their important work are explained in Bellman’s book [1].

Equation (1.2) is said to be superlinear if \( \gamma > 1 \) and sublinear if \( \gamma < 1 \). Properties of the equation in the two cases differ although some kind of duality principle seems to exist, observed particularly in the theory of oscillation by Coffman and Wong [8].

Wong’s exhaustive survey article [31] covers a wide spectrum of interesting results for equation (1.2) with a general coefficient \( q(t) \). In this paper, however, we assume that the coefficient \( q(t) \) is non-negative:

\[ q(t) \geq 0, \quad t > 0. \tag{1.3} \]

Equation (1.1) has been further generalized to

\[ y'' + yF(y^2, t) = 0, \tag{1.4} \]

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where $F$ satisfies

$$F(z,t) > 0 \quad \text{and} \quad \frac{\partial F(z,t)}{\partial z} > 0 \quad \text{for} \quad z > 0, \ t > 0,$$

and has been studied by many authors, notably Nehari, Coffman, and Wong. In this paper, we restrict ourselves to equation (1.1).

In recent years, there has been a revived interest in (1.1) in connection with nonlinear reaction-diffusion equations. The works of Gidas, Ni, and Nirenberg [9, 10] point out that radially symmetric solutions play a very important role.

Let $u(r) = u(|x|)$ be a radially symmetric solution of the semilinear elliptic equation

$$\Delta u + g(r)f(u(r)) = 0, \quad u \in \Omega \subset \mathbb{R}^n,$$

where $\Omega$ can be the whole space $\mathbb{R}^n$, a ball of radius $b$, or the annulus region between two concentric balls of radii $a$ and $b$, respectively. The function $u$ satisfies the ordinary differential equation

$$u'' + \frac{n-1}{r} u' + g(r)f(u) = 0, \quad u \in [a, b],$$

which is equivalent to the equation

$$(r^{n-1}u')' + r^{n-1}g(r)f(u) = 0.$$  

With the change of variables $t = r^{2-n}$ (for $n \neq 2$) or $t = \log r$ (for $n = 2$), and $y(t) = u(r(t))$, (1.7) reduces to an equation of the form (1.1). We therefore study, alongside with equation (1.1), the equation

$$u'' + p(t)u' + f(u) = 0,$$

with particular attention to the case when $p(t) = (n - 1)/t$.

Many authors have investigated boundary value problems either in a bounded or semi-bounded interval. The boundary conditions involved are generally of the Dirichlet or Neumann type. In case of a semi-bounded interval, the asymptotic condition $y(t) \to 0$ as $t \to \infty$ is usually assumed. If a non-negative solution satisfying the given boundary conditions exists, it is called a ground state.

Among the interesting questions, a difficult one is the uniqueness problem for the ground state solution. Traditional approaches, for example those expounded in Bernfeld and Lakshmikantham [4], seem to be of no avail.

In the sublinear case, however, the uniqueness of the ground state is well known. It dates back to Picard [29], and similar theorems for a class of integral equations have been obtained by Urysohn [30], later generalized by Krasnosel’skii and Ladyzenskii [16]. Krasnosel’skii’s book [15] is the only reference to these results available in English. We shall see in Section 3 how to establish the result for (1.1) easily using the Sturm comparison theorem.