On the uniqueness of conduction in relaxation-case semiconductors

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1. Introduction

In this paper we consider the conduction in a relaxation-case semiconductor. This problem leads to a system of three nonlinear differential equations together with boundary conditions (see [3], for instance). The question has come up whether a certain simplified version of this system already allows an explanation of the phenomenon of “switching” in semiconductors, see [1], [2]. Hence it is of particular interest whether the simplified model allows the existence of multiple solutions. Numerical experiences (see [1]) suggest that the simplified system is uniquely solvable. In this paper we give a rigid proof that this conjecture is correct, thereby showing that the simplified system is not an adequate model to explain “switching”. Whether this is also the case for the original system is as yet unknown. The method of proof presented here does not apply in the unsimplified case.

2. The uniqueness result

Let us consider the nonlinear system

\[ \Psi''(x) = 8 \pi^2 n_i e^{-1} \left\{ \sinh (\beta \Phi_0(x) - \beta \Psi_0(x)) - \sinh (\beta \Phi(x) - \beta \Psi(x)) \right\}, \quad 0 \leq x \leq L, \]

\[ \Phi'(x) = j \left( \cosh (\beta \Phi(x) - \beta \Psi(x)) \right)^{-1}, \quad 0 \leq x \leq L, \]

\[ \Phi(L) - \Phi(0) = U, \quad \Phi(0) - \Psi(0) = \alpha_1, \quad \Phi(L) - \Psi(L) = -\alpha_2. \]

Here \( \beta > 0, n_i > 0, \epsilon > 0, j \neq 0, U \neq 0, L \neq 0, \alpha_1 \neq 0 \) and \( \alpha_2 \neq 0 \) stand for physical constants, \( \Phi_0, \Psi_0 \in C[0, L] \) are given. System (1), (2) is a simplified version of the differential system governing the relaxation regime in a semiconductor, see [1], [2].

We note that together with \( (\Psi, \Phi) \) also the pair \( (\Psi + \gamma, \Phi + \gamma), \gamma \in \mathbb{R} \), solves (1), (2). Hence we add the normalization

\[ \Phi(0) = 0. \]

In physical terms this merely means a translation of the coordinate system.

In the sequel we shall use the notations

\[ c := 8 \pi^2 n_i \cdot e^{-1}, \quad q(x) := \sinh (\beta \Phi_0(x) - \beta \Psi_0(x)), \quad 0 \leq x \leq L. \]

Note that \( c > 0 \). We obtain the following uniqueness result:

**Theorem 1.** For every \( j \neq 0 \), the system (1), (2), (3) has at most one solution pair \( (\Psi, \Phi) \).
Proof. Let \((\Psi, \Phi)\) be any solution of (1), (2), (3). Then we have \(\Phi' \in C^1 [0, L]\), and differentiation yields for \(x \in [0, L]\)

\[
\Phi''(x) = -\frac{j \sinh (\beta \Phi(x) - \beta \Psi(x))}{\cosh^2 (\beta \Phi(x) - \beta \Psi(x))} \beta (\Phi'(x) - \Psi'(x)).
\]  

(5)

Subtraction of (5) from the first equality in (1) shows that \(y := \Phi - \Psi\) solves the second-order boundary value problem

\[
y''(x) = -\frac{j \sinh (\beta y(x))}{\cosh^2 (\beta y(x))} \beta y'(x) - c q(x) + c \sinh (\beta y(x)), \quad 0 \leq x \leq L.
\]

(6)

\[
y(0) = \alpha_1, \quad y(L) = -\beta_2.
\]

(7)

Now let \((\Psi_1', \Phi_1), (\Psi_2', \Phi_2)\) be two different solutions of (1), (2), (3), and let \(y_1, y_2\) denote the corresponding solutions of (6), (7). We show that \(y_1 = y_2\). One this is proved we can complete the proof as follows:

From (1) we have on \([0, L]\):

\[
\Phi_1'(x) = j \{\cosh (\beta y_1(x))\}^{-1} = j \{\cosh (\beta y_2(x))\}^{-1} = \Phi_2'(x),
\]

and (3) shows that \(\Phi_1 = \Phi_2\). Hence \(\Psi_1 = \Phi_1 - y_1 = \Phi_2 - y_2 = \Psi_2\).

We thus have to show that \(y_1 = y_2\). Assume that \(y_1 \neq y_2\). Since \(y_1\) and \(y_2\) are different solutions of the same second order differential equation, namely (6), by the standard uniqueness theorem they must have different initial conditions. However, from (7) we have \(y_1(0) = y_2(0)\). It follows that \(y_1'(0) \neq y_2'(0)\).

We thus may assume that \(y_1'(0) > y_2'(0)\). Then there exists a smallest \(x_0 \in (0, L]\) such that \(y_1(x_0) = y_2(x_0)\).

Hence \(y_1(x) > y_2(x)\) in \((0, x_0]\), and we have

\[
\Phi_1'(0) = j \{\cosh (\beta y_1(0))\}^{-1} = j \{\cosh (\beta y_2(0))\}^{-1} = \Phi_2'(0),
\]

and

\[
\Phi_1'(x_0) = j \{\cosh (\beta y_1(x_0))\}^{-1} = j \{\cosh (\beta y_2(x_0))\}^{-1} = \Phi_2'(x_0).
\]

Thus we have

\[
0 = \Phi_1'(x_0) - \Phi_2'(x_0) - (\Phi_1'(0) - \Phi_2'(0)) = \int_0^{x_0} (\Phi_1''(s) - \Phi_2''(s)) \, ds,
\]

and (5) implies

\[
0 = \int_0^{x_0} \left[ -\frac{j \sinh (\beta y_1(s))}{\cosh^2 (\beta y_1(s))} \beta y_1'(s) + \frac{j \sinh (\beta y_2(s))}{\cosh^2 (\beta y_2(s))} \beta y_2'(s) \right] \, ds.
\]

(8)

Finally, we obtain from (6)

\[
y_1'' - y_2'' = \frac{j \sinh (\beta y_2)}{\cosh^2 (\beta y_2)} \beta y_2' - \frac{j \sinh (\beta y_1)}{\cosh^2 (\beta y_1)} \beta y_1' + c (\sinh (\beta y_1) - \sinh (\beta y_2)),
\]

and (8) yields with \(c > 0, \beta > 0\):

\[
y_1'(x_0) - y_2'(x_0) - (y_1'(0) - y_2'(0)) = c \int_0^{x_0} (\sinh (\beta y_1(s)) - \sinh (\beta y_2(s)) \, ds > 0,
\]

since \(y_1(s) > y_2(s)\) in \((0, x_0]\).

Consequently, we have \(y_1'(x_0) - y_2'(x_0) > y_1'(0) - y_2'(0) > 0\) which contradicts the assumption \(y_1(s) > y_2(s), s \in (0, x_0]\).

q.e.d.