Constrained normalization of Hamiltonian systems and perturbed Keplerian motion

By Jan-Cees van der Meer, Center for Mathematics and Computer Science (CWI), Kruislaan 413, 1098 SJ Amsterdam, and Richard Cushman, Mathematics Institute, Rijksuniversiteit Utrecht, Budapestaan 6, 3584 CD Utrecht, The Netherlands.

I. Introduction

In this paper we develop a theory for normalizing constrained Hamiltonian systems. We make use of some ideas of Moser [6] concerning constrained Hamiltonian systems (see also [2]). The idea of constrained normalization is the following. Consider a Hamiltonian system with Hamiltonian function $H$ on $(\mathbb{R}^{2n}, \omega)$, where $\omega$ is the standard symplectic form. Denote such a system by $(H, \mathbb{R}^{2n}, \omega)$.

For a symplectic submanifold $M \subset \mathbb{R}^{2n}$ define the constrained system corresponding to $(H, \mathbb{R}^{2n}, \omega)$ by $(H|_M, M, \omega|_M)$. Here $|_M$ means restriction to $M$.

We give a normalization algorithm for the system $(H, \mathbb{R}^{2n}, \omega)$ which on $M$ restricts to a normalization of the constrained system. The advantage is that the necessary computations are performed in the ambient space $\mathbb{R}^{2n}$, where they are easier to do.

The paper is organized as follows. In the second section we give the facts about constrained Hamiltonian systems needed for the development of the constrained normalization algorithm in section three. In the fourth section we introduce the Kepler system on $\mathbb{R}^{2n}$. As is well known (see [5]) the Kepler system, after regularization, can be considered as a system on $\mathbb{R}^{2n+2}$ constrained to $T^+ S^n$, the cotangent bundle to the n-sphere minus its zero section. The same techniques enable us to consider perturbed Kepler systems as constrained systems, as is shown in section five. The facts proved in section four show that we may apply the constrained normalization algorithm to perturbed Keplerian systems. We illustrate this with two examples: (i) the lunar problem (section six), and (ii) the main problem of artificial satellite theory (section seven). The treatment of the main problem takes as its starting point the results of Deprit [3] concerning the elimination of the parallax. The normalization up to second order of the lunar problem provides a straightforward alternative for the quite different approach of Kummer [4].
2. Constrained Hamiltonian systems

Consider $\mathbb{R}^{2n}$ with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ and standard symplectic form $\omega(x, y) = \sum_{i=1}^{n} dx_i \wedge dy_i$. For $m < n$ let $F_1, \ldots, F_{2m} \in C^\infty(\mathbb{R}^{2n})$ be such that $dF_1, \ldots, dF_{2m}$ are independent on $M = \{(x, y) \in \mathbb{R}^{2n} | F_1(x, y) = \cdots = F_{2m}(x, y) = 0\}$, that is, $M$ is a smoothly embedded submanifold of $\mathbb{R}^{2n}$. Furthermore suppose that the matrix $C = (c_{ij} = \{F_i, F_j\})$ is nonsingular at every point of $M$. Then $M$ is a symplectic manifold with symplectic form $\omega|_M$, the restriction of the symplectic form $\omega$ to $M$.

For $H \in C^\infty(\mathbb{R}^{2n})$ the restriction of the Hamiltonian vector field $X_H$ to $M$ need not be tangential to $M$. However we can construct a vector field tangential to $M$ by considering $X_{H|_M}$ on $(M, \omega|_M)$, where $H|_M$ is the restriction of $H$ to $M$. We call $X_{H|_M}$ the constrained Hamiltonian vector field corresponding to $H$.

Another way to describe the constrained vector field is that $X_{H|_M}$ is the image of the projection of $X_H$ on $T_M\mathbb{R}^{2n}$ with respect to the splitting of $T\mathbb{R}^{2n}$ into $T_M\mathbb{R}^{2n}$ and its $\omega$-orthogonal complement.

Let $\mathcal{I}$ be the ideal of $C^\infty(\mathbb{R}^{2n})$ generated by $F_1, \ldots, F_{2m}$, that is, $\mathcal{I}$ is the ideal of functions vanishing on $M$. Furthermore let $L_H$ denote the derivative defined by $L_H = \{., H\}$, where $\{., \}$ is the Poisson bracket on $\mathbb{R}^{2n}$ with respect to the symplectic form $\omega$.

**Lemma 1.** The following statements are equivalent:

(i) $X_{H|_M} = X_H$ on $M$.
(ii) $\{H, F_j\} \in \mathcal{I}$, for $j = 1, \ldots, 2m$.
(iii) $(\exp L_H)(\mathcal{I}) \subseteq \mathcal{I}$.
(iv) $M$ is an invariant manifold of $X_H$.
(v) $X_H$ is tangent to $M$ at each point of $M$.

*Proof.* The proof is easy and left to the reader. $\square$

Let $H \in C^\infty(\mathbb{R}^{2n})$. When $X_H$ is not tangent to $M$ we can construct a function $H$ such that $H|_M = H|_M$, $X_H$ is tangent to $M$, and $X_{H|_M} = X_{H|_M}$. The construction of $H$ is given in Lemma 2. Note that $H$ need not be a smooth function on all of $\mathbb{R}^{2n}$. In fact $H$ is first constructed on $M$ and then extended to some open neighborhood of $M$ in $\mathbb{R}^{2n}$. Let $C^{-1} = (c^{ij})$ be the inverse of the matrix $C$.

**Lemma 2.** If $H = H + \sum_{i=1}^{2m} a_i F_i$, with $a_i = \sum_{j=1}^{2m} c^{ij} \{H, F_j\}$, then $X_{H|_M} = X_H$ on $M$. 