COMMENTS ON ‘OUTWARD TRANSPORT OF ANGULAR MOMENTUM...’ BY E. M. DROBYSHEVSKI

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(Received 16 March; in revised form 18 July, 1978)

Abstract. Drobyshevski’s effect for maintaining the equatorial acceleration of the Sun is characterized to be of order \( \lambda/R \) where \( \lambda \) represents the eddy size and \( R \) the solar radius. The weakness of this mechanism is, therefore, not surprising.

Furthermore we have discussed the idea of replacing the two-dimensional eddy tubes by spherical eddy configurations. We can state here that a theory which is linear in the rotational rate \( \Omega \) and uses the radial direction as the only source (besides \( \Omega \)) of anisotropy cannot explain the observed phenomenon of differential rotation.

1. As is well established knowledge, the hydrodynamic basis of turbulence theories of differential rotation is given by the Reynolds equation

\[
\frac{\nu}{r^2} \frac{\partial}{\partial r} \left( r^4 \frac{\partial f}{\partial r} \right) + \frac{\nu}{\sin^3 \theta} \frac{\partial}{\partial \theta} \left( \sin^3 \theta \frac{\partial f}{\partial \theta} \right) = \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial r} (r^3 Q_{re}) + \\
+ \frac{1}{\sin^3 \theta} \frac{\partial}{\partial \theta} (\sin^2 \theta Q_{\theta \phi}),
\]

where any meridional circulation is neglected. \( \nu \) is the molecular viscosity or, what is more likely, the eddy viscosity of a small-scaled turbulence. \( \Omega \) is the angular velocity of global rotation and the \( Q \)-quantities are cross-components of the correlation tensor \( Q_{ij} = u'_i(x, t)u'_j(x, t) \). Per definition, \( Q_{ij} \) is not a pseudo-tensor and is symmetric with respect to its indices. \( Q_{re} \) represents the radial and \( Q_{\theta \phi} \) the latitudinal flux of angular momentum.

The phenomenon of differential rotation may be explained as a non-constant solution \( \Omega(r, \theta) \) of the Reynolds Equation (1). Apart from special artificial boundary conditions such a solution only exists in case the components \( Q_{re} \) and \( Q_{\theta \phi} \) do not vanish with vanishing derivatives of \( \Omega \). I.e., a theory of differential rotation based on turbulence requires the turbulent fluxes of angular momentum to go essentially with \( \Omega \) instead with its derivatives. We shall denote those parts of \( Q_{ij} \) which are not ‘proportional’ to derivatives of \( \Omega \) by \( \tilde{Q}_{ij} \). The difference between \( Q_{ij} \) and \( \tilde{Q}_{ij} \), vanishing for rigid rotation, only slightly modifies the left-hand side of (1).

As two different quantities, namely \( Q_{re} \) and \( Q_{\theta \phi} \), can be responsible for producing a differential rotational law, also two different models for it can exist. Gilman has started his calculations by favouring the \( \tilde{Q}_{\theta \phi} \)-correlation (see Starr and Gilman, 1965) whereas other theoretical suggestions led to emphasising the \( Q_{re} \)-correlations (Rüdiger, 1977).

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In more complete theoretical investigations, of course, both quantities prove to be of importance (see Gilman, 1977).

2. In a recent paper Drobyshevski (1977) has proposed a $Q_{\alpha\phi}$-mechanism for maintaining solar equatorial acceleration. It is based on the suggestion that oppositely rotating eddy tubes have different buoyancy in a rotating star. The quantity $V_r\omega$ proves to be always positive where $\omega$ represents the rigid eddy rotation and $V_r \sim g\Omega$ its radial movement caused by the joint action of global rotation and gravity $g$. The eddy tubes are assumed to be two-dimensional ($\partial/\partial z = 0$) with respect to the axis of rotation. The resulting turbulent flux of momentum seems to be linear in $\Omega$ and leads to only 3% differential rotation $\partial\Omega/\partial r$ for a convection zone of about 30% of solar radius. He presumes the small effect from his mechanism to be a consequence of the estimative character of his calculations.

The following considerations shall be based on the assumption that rotating eddy tubes really possess such characteristic radial movements $V_r$. Restricting ourselves to the equatorial plane only – and that is what Drobyshevski does – we find the geometrical situation presented in Figure 1. $P_3$ marks the fixed point of the wanted correlation measurement. $P_1$ denotes the axis of global rotation and $P_2$ the stochastically placed axis of an eddy tube. Hence,

\begin{align*}
u'_r &= l\omega \sin \phi + V_r \cos \psi, \\
u'_\phi &= l\omega \cos \phi + V_r \sin \psi,
\end{align*}

with $\sin \psi/\sin(\phi + \psi) = l/r$. From the latter relation follows $\psi \approx l \sin \phi/r$ in the case of $l/r$ small enough. Then the leading term of the velocity correlation is

\begin{equation}
u'_r\nu'_\phi = \frac{l^2 \omega V_r}{r} \sin^2 \phi + l\omega V_r \cos \phi.
\end{equation}

Fig. 1. The equatorial plane with the axis, $P_2$, of a cylindrical eddy tube. $P_3$ fixes the point of measurement whereas $P_1$ denotes the axis of global rotation.