The self-similar solutions to a fast diffusion equation

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1. Introduction

In this paper we study in $\mathbb{R}^n$ the equation

$$\Delta u^\alpha - \frac{x \cdot \nabla u}{2} + \mu u + u^\beta = 0, \quad (1)$$

where $0 < \alpha < 1$, $\beta > 1$ and $\mu > 0$ is a parameter. The equation arises from the study of blow-up self-similar solutions of the heat equation

$$\psi_t = \Delta \psi^\alpha + \psi^\beta. \quad (2)$$

In fact, let

$$\psi(y, t) = (T - t)^{-1/(\beta - 1)}(2m)^{1/(\beta - 1)}u(x), \quad (3)$$

where

$$m = \frac{\beta - \alpha}{2(\beta - 1)},$$

$$x = \frac{(2m)^m y}{(T - t)^m}. \quad (4)$$

Then it is easy to verify that $u$ satisfies (1) with

$$f(u) = -\mu u + u^\beta, \quad \mu = \frac{1}{\beta - \alpha}.$$  

The fast diffusion equation (2) arises from many applications in plasma physics and chemical engineering. In this paper we always assume that $f(u)$ takes the above form with $\mu > 0$ as a free parameter. The main purpose of our study is to establish the existence (non-existence) of positive ground state of (1) which, in the case $\mu = 1/(\beta - \alpha)$, plays a role of demonstrating

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the power law of the heat conduction process of which (2) models. Then the result can be applied to characterize the local behaviour of solutions of (2) near a blow-up point (see [3], [4], [5], [8], [10] and [14]). In particular, we have the following result on the existence and non-existence of self-similar solutions of (2).

**Theorem 1.** Let $0 < \alpha < 1$ and $\beta > 2 - \alpha$.

(i) For $\beta/\alpha < \infty$ when $n = 1, 2$ or $\beta/\alpha < (n + 2)/(n - 2)$ when $n \geq 3$ there exists no non-constant positive radial symmetric self-similar solution of (2).

(ii) For $(n + 2)/(n - 2) < \beta/\alpha < \tilde{p}$ when $n \geq 3$ there exist an infinite number of positive radial symmetric self-similar solution of (2).

Here $\tilde{p}$ is defined as

$$
\tilde{p} = \begin{cases} 
\infty & n \leq 10 \\
\frac{(n - 2)^2 - 4n + 8\sqrt{n - 1}}{(n - 2)(n - 10)} & n > 10.
\end{cases}
$$

A direct consequence of the above theorem is the following result on characterizing local behaviour of solutions of (2) near a blow-up point.

**Theorem 2.** Let $0 < \alpha < 1$ and $\beta > 2 - \alpha$. Let $\beta/\alpha < \infty$ when $n = 1, 2$ or $\beta/\alpha < (n + 2)/(n - 2)$ when $n \geq 3$. Let $\phi(x, t)$ be a radially symmetric solution of (2) which blows up at point $x = 0$ with blow-up time $T > 0$. Then

$$
\lim_{t \to T} \phi(x, t)(T - t)^{1/(\beta - 1)} \to \left(\frac{1}{\beta - 1}\right)^{1/(\beta - 1)}
$$

uniformly for $|x| < C(T - t)^{\frac{1}{\beta - 1}}$.

**Proof.** It follows directly from Theorem 1 and an argument in [10].

For simplicity, we shall only consider radial solutions of (1). In this case we can write $u(x) = u(r)$, where $r = |x|$. Let $u^w(r) = w^{(\alpha - m)r}\alpha^{-(\beta - 1)}$, then

$$
w'' + \frac{n - 1}{r} w' - \left(\frac{r}{2} w' + \lambda w\right)w^q + w^p = 0,
$$

where $p = \frac{\beta}{\alpha}, q = \frac{1 - \alpha}{\alpha}$ and $\lambda = \alpha\mu$. The case corresponding to self-similar solution of (2) is

$$
\lambda = \frac{1}{p - 1} = \frac{\alpha}{\beta - \alpha}.
$$

Next we give a simple definition.