ON DEFINING THE SPIN MOMENT
IN THE TETRAD THEORY
OF GRAVITATION

V. N. Tunyak

A general definition of the spin moment is presented in the tetrad formulation of the relativistic theory of gravitation; it is based on the conditions for the invariance of the corresponding action integral relative to infinitesimal tetrad transformations (the so-called tetrad spin moment) and infinitesimal coordinate transformations (the so-called coordinate spin moment). It is shown that the tetrad formulation of the general theory of relativity (TFGTR) and the tetrad theory of gravitation (TTG) in a space of absolute parallelism lead to fundamentally different definitions of spin, since in the Riemannian geometry of the TFGTR only the coordinate spin moment is physically meaningful, whereas in the space of absolute parallelism of the TTG only the tetrad spin moment has essential signification. It is also indicated that the Pellegrini–Plebanski theory (PPT) leads to an unsatisfactory hybrid definition of spin in the form of the coordinate spin moment of the gravitational and boson fields and the tetrad spin moment of the gravitational and fermion fields, the gravitational field entering into these spin moments of the PPT with opposite signs.

A general formulation of the weak law of conservation was presented earlier [1, 2] within the framework of the tetrad theory of gravitation (TTG) in a space of absolute parallelism \( W_t \); this theory takes the tetrad components \( \mathbf{h}(\mathbf{\xi}) \) as the 16 dynamic potentials of the gravitational field. A consideration of this weak law of conservation in the important particular case of arbitrary infinitesimal coordinate transformations

\[
\mathbf{x}'^a = \mathbf{x}^a + \mathbf{\alpha}^a
\]

leads [1, 2] to a fundamentally interesting definition of the sources of the tetrad gravitational field, in the form of the total canonical energy-momentum tensor of nongravitational matter; this of course plays a fundamental part in the special-relativistic theory of energy-momentum localization [3]. The aim of the present investigation lies in making a further study of the weak law of conservation of the TTG [1, 2] for infinitesimal coordinate transformations of a special type

\[
\mathbf{x}'^a = \mathbf{x}^a + \mathbf{\alpha}^a, \quad \mathbf{\alpha}^a = \mathbf{\alpha}^a_{[\mathbf{\xi}],} \quad \mathbf{\partial}_a \mathbf{\alpha}^a = 0
\]

and infinitesimal rigorous tetrad transformations

\[
\mathbf{\partial} \mathbf{h}(\mathbf{\xi}) = \mathbf{b}(\mathbf{\xi}) \mathbf{\partial} \mathbf{h}(\mathbf{\xi}), \quad \mathbf{b}(\mathbf{\xi}) + \mathbf{b}(\mathbf{\xi}) = 0, \quad \mathbf{\partial}_\mathbf{\alpha} \mathbf{b}(\mathbf{\xi}) = 0,
\]

leading to two fundamental definitions of the spin moment in the TTG in the form of the so-called coordinate and tetrad spin moments respectively.

In solving this problem we shall start from the single variational principle of the TTG [1, 2]

\[
\mathbf{\varepsilon} \int (L_m + L_h) \mathbf{V} - \mathbf{\mu} \mathbf{d}^a x = 0
\]

for a certain self-consistent set of gravitational and boson fields described by tensor potentials \( \mathbf{Q}_A \) (A, B, C, ... are the collective tensor indices), and fermion fields represented by Dirac bispinors \( \mathbf{\Psi} \) and \( \mathbf{\bar{\Psi}} \). The expressions \( L = L_m + L_h, \ L_m, \ L_h \) respectively represent the total Lagrangian, the gravitational Lagrangian of the TTG [1, 2], and the Lagrangian of nongravitational matter

\[
L_m = L_m(h^{(\xi)}_\mathbf{\alpha}, \mathbf{Q}_A, \mathbf{\Psi}, \mathbf{\partial}_\mathbf{\alpha} \mathbf{\Psi}, \mathbf{\bar{\Psi}}, \mathbf{\partial}_\mathbf{\alpha} \mathbf{\bar{\Psi}}),
\]

constituting a direct covariant generalization of the special-relativistic expression $L_m$ to the space $W_4$ \[1, 2\]. The covariant derivative of the tensor potential $Q_A$ in $W_4$ has the form

$$\nabla \cdot Q_A = \partial_\alpha Q_A - \gamma^\alpha_{\beta\lambda} Q_B f_B^{\beta\lambda},$$

(6)

where $\gamma^\alpha_{\beta\lambda} = h^{(\alpha}_{(\beta} \partial_\lambda h^{(\beta)}_{\lambda)}$ is the connectedness (compendency) of the absolute parallelism; $f_B^{\beta\lambda}$ are certain combinations of Kronecker symbols $\delta^\mu_{\mu}$ depending on the tensor rank of $Q_A$. The general formulation of the weak law of conservation corresponding to this variational principle has the form

$$\nabla^\mu \left[ e^\nu L - \frac{\partial L_m}{\partial (\nabla \cdot Q_A)} \delta Q_A + \frac{\partial L_m}{\partial (\nabla \cdot \Psi)} \delta \Psi + \frac{\partial L_m}{\partial (\nabla \cdot \bar{\Psi})} \delta \bar{\Psi} + \delta h^{(\alpha}_{\alpha}) \times \right] \times \left[ U_{\beta\lambda}^{\mu\nu} \frac{\partial L_m}{\partial (\nabla \cdot Q_A)} Q_B h^{(\alpha}_{(\beta} f_B^{\beta\lambda}) \right] = 0,$$

(7)

where $\delta$ is the form variation of the function; $\nabla^\mu$ is the covariant derivative relative to the Cristoffel connection derived from the Riemannian metric $g_{\mu\nu} = h^{(\mu}_{(\mu} h^{(\nu)}_{\nu)} \eta_{(\nu)(\nu)}$, $\eta_{(\nu)(\nu)} = \text{diag} (1, -1, -1, -1)$ is the local Minkowski metric

$$U_{\beta\lambda}^{\mu\nu} = \partial L_h / \partial (\nabla \cdot Q_A).$$

(8)

the so-called superpotential [1, 2].

If we substitute the potential variations due to the infinitesimal coordinate transformation (2)

$$\delta h^{(\alpha}_{\alpha} = a_{\alpha\beta} x^\beta \left( g^{\beta\gamma} \partial_\gamma h^{(\alpha}_{\beta} + h^{(\alpha}_{\beta} \partial_\beta g^{\beta\gamma} \right) + h^{(\alpha}_{\alpha} a_{\mu\nu},$$

(9)

$$\delta Q_A = a_{\alpha\beta} Q_B f_B^{\beta\lambda} g^{\gamma\nu} + a_{\alpha\beta} x^\lambda \left( g^{\gamma\nu} \delta_\lambda Q_A + Q_B f_B^{\beta\lambda} \delta_\beta g^{\gamma\nu} \right),$$

(10)

$$\delta \Psi = a_{\alpha\beta} x^\lambda \delta_\lambda \Psi,$$

(11)

in the weak law of conservation (7) we obtain the following law of conservation of the total (orbital $P^\mu \sigma^\nu$ and coordinate spin $V^\mu \sigma^\nu$) moment:

$$\partial \omega \left[ V^\mu - g [ P^{\mu\nu} + V^{\mu\nu}] \right] = 0,$$

(12)

where

$$V^{\mu\nu} = 2U^{\mu\nu} \eta_{\nu\lambda}.$$

(13)

In order to determine the tetrad spin moment $W(\mu)(\nu)^{\lambda}$ we substitute the potential variations due to the infinitesimal tetrad transformation (3)

$$\delta Q_A = 0, \quad \delta \Psi = \frac{i}{4} b^{(\mu\nu)} \sigma_{(\mu)(\nu)} \Psi,$$

(14)

into Eq. (7) where $\sigma_{\mu\nu} = i \gamma_{[\mu \nu]}$ is the so-called matrix spin tensor $\gamma_{\mu\nu} = h^{(\mu}_{(\mu} \gamma_{(\nu)}^{(\nu)}$, $\gamma_{\nu}$ are certain special-relativistic Dirac matrices. We accordingly arrive at the tetrad spin moment, conserved in accordance with the equation

$$\nabla \cdot W_{(\mu)(\nu)^{\lambda}} = 0,$$

(15)

of the gravitational field and nongravitational matter

$$W_{(\mu)(\nu)^{\lambda}} = 2U_{(\mu)(\nu)^{\lambda}} + S_{(\mu)(\nu)^{\lambda}},$$

(16)

where

$$S_{(\mu)(\nu)^{\lambda}} = \frac{2}{\partial (\nabla \cdot Q_A)} \left[ Q_B f_B^{\beta\lambda} g_{\beta\nu}^{\gamma\nu} + \frac{i}{2} \sigma_{(\mu)(\nu)} \left( \frac{\partial L_m}{\partial (\nabla \cdot \Psi)} - \Psi \frac{\partial L_m}{\partial (\nabla \cdot \bar{\Psi})} \right) \right]$$

(17)

is the spin moment of nongravitational matter. Comparison between the resultant expressions for the coordinate and tetrad spin moments shows that, within the framework of the TTG, only the definition of the tetrad spin moment generalizing the special-relativistic definition of spin (and transforming into the latter when the gravitational field is excluded) remains physically consistent in $W_4$ space, while the coordinate spin moment (13), which does not contain the corresponding contribution of nongravitational matter, and furthermore possesses the opposite sign to the tetrad spin moment of the gravitational field obtained from (16), has no physical meaning. It is also an important fact that, according to (12) and (15), the principle of correspondence with the special-relativistic law of conservation of the total moment

$$\partial \omega \left[ P^{\mu\nu} + S^{\mu\nu} \right] = 0,$$

(18)