A CHARACTERIZATION OF SUBSYSTEMS IN PHYSICS

DIRK AERTS and INGRID DAUBECHIES*

Theoretische Natuurkunde, Vrije Universiteit Brussel, Pleinlaan 2,
B-1050 Brussel, Belgium

ABSTRACT. Working within the framework of the propositional system formalism, we use a previous study [1] of the description of two independent physical systems as one big physical system to derive a characterization of a (non-interacting) physical subsystem. We discuss the classical case and the quantum case.

1. INTRODUCTION

We shall follow Piron [2] and describe any physical system by means of the collection of its properties, or, equivalently, of the yes-no experiments which can be carried out on this system. In [2], it is shown that this collection is a propositional system, that is a complete, orthocomplemented, weakly modular, atomic lattice satisfying the covering law. The states of the physical system are represented by the atoms of the lattice. For the definitions of these concepts and the physical justification of this approach, see [2] or also [1]. In what follows we shall use the abbreviation PROP for these propositional systems.

In [1], we studied the description of two non-interacting physical systems as one joint physical system. We denote these two independent systems by $S_1$, $S_2$, and the big physical system containing them both by $S$. The corresponding PROP's are $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}$. From a few simple arguments resulting from physical considerations, we arrived at the following structure (see [1], §2):

(1.1) There exist $c$-morphisms $h_1$, $h_2$ from $\mathcal{L}_1$, $\mathcal{L}_2$ to $\mathcal{L}$ with $h_1(I_1) = 1$, $h_2(I_2) = 1$. ($I_1$, $I_2$ are the maximal elements in $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}$, respectively.) This is the mathematical translation of the fact that the structures of $S_1$ and of $S_2$ are conserved.

(1.2) For $a_1$ in $\mathcal{L}_1$, $a_2$ in $\mathcal{L}_2$, we have $h_1(a_1) \land h_2(a_2)$, ($h_1(a_1)$ and $h_2(a_2)$ are compatible). This is the mathematical formulation of the fact that $S_1$, $S_2$ are supposed to be independent.

(1.3) For $p_1$ atom in $\mathcal{L}_1$, $p_2$ atom in $\mathcal{L}_2$, we have that $h_1(p_1) \land h_2(p_2)$ is an atom in $\mathcal{L}$. This means that maximal information on $S_1$, $S_2$ separately yields maximal information on $S$: $S_1$, $S_2$ are the only constituents of $S$.

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†A $c$-morphism is a map conserving the complete, orthocomplemented, weakly modular lattice structure. When a $c$-morphism maps the maximal element onto the maximal element it is said to be unitary (see [2]).

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With these three requirements we were able in [1] to prove some results about the PROP of the joint physical system, and this in both quantum and classical cases. Indeed, both classical and quantal systems can be described in the framework of the propositional approach. For the classical systems, one introduces one more property, namely distributivity, representing the well-known physical fact that in this case all possible experiments can be carried out independently of each other [2]. Such a distributive PROP can be shown to be isomorphic to \( \mathcal{P}(\Omega) \), i.e. to the lattice (with respect to set-theoretic inclusion) of all the subsets of the set \( \Omega \) of its atoms [2]. This set \( \Omega \) is then called the phase space of the classical system. Using the three conditions mentioned above (in fact only the first and the third ones: the second one becomes redundant in this case) we proved that when a classical physical system \( S \) is constituted by two classical systems \( S_1, S_2 \) with respective phase spaces \( \Omega_1, \Omega_2 \), its phase space is given by \( \Omega_1 \times \Omega_2 \) [1].

When the PROP is not distributive, we make a distinction between pure quantum systems and more general systems. A pure quantum system has no classical features, i.e. no superselection rules: there does not exist any yes—no experiment compatible with all the others. When this is the case, we say the PROP is irreducible. It is proven in [3] that any such irreducible PROP (granted that it contains at least four orthogonal atoms) is isomorphic to the lattice of all biorthogonal subspaces of some vectorspace \( V, \mathbb{K} \), where the orthogonality is defined with respect to some sesquilinear form on \( V, \mathbb{K} \), and where \( F^\perp + F^\perp = V \) for any subspace \( F \) of \( V \). This structure looks quite formidable, but it is not really so terrifying. If one takes the field \( \mathbb{K} \) to be \( \mathbb{C} \), one can prove [3], [4] that the structure is exactly the one encountered in the usual quantum formalism: the lattice described above becomes now the lattice of all closed subspaces of a complex Hilbert space \( \mathcal{H} \), or, equivalently, the lattice of all projection operators in this Hilbert space. The atoms (= states) are then given by the one-dimensional subspaces of \( \mathcal{H} \).

When there exist superselection rules, the PROP can be considered as a combination of pure quantal propositional systems [2]. We will not consider such composite systems here.

Applying our three conditions stated above to this setting, we proved in [1] that when a physical quantum system \( S \) is made up of two pure quantum systems \( S_1, S_2 \) with respective Hilbert spaces \( \mathcal{H}_1, \mathcal{H}_2 \), it is described by the Hilbert space \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) or \( \mathcal{H}_1^+ \otimes \mathcal{H}_2^+ \). Since our three conditions proved to be sufficient to derive the usual coupling procedures for the simultaneous description of two independent physical systems, it is a natural question to ask whether they can also be used to characterize physical subsystems of a big physical system. How this is done will be explained in the next section.

2. CHARACTERIZATION OF A PHYSICAL SUBSYSTEM

Our aim is here to investigate the conditions under which a sublattice \( \mathcal{L} \) of the PROP \( \mathcal{L} \) of a physical system \( S \) can be considered as the PROP of a physical subsystem \( \tilde{S} \) of \( S \). In other words, given \( \mathcal{L} \subset \mathcal{L} \), we want to be able to ascertain whether \( L_1, L_2, h_1, h_2 \) exist, satisfying conditions (1.1), (1.2), (1.3) and for which

\[
h_1(L_1) = \tilde{L}.
\]

This motivates the following definition: