THE BAKER–CAMPBELL–HAUSDORFF FORMULA FOR THE CHIRAL 
SU(2) SUPERGROUP

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ABSTRACT. In the theory of supersymmetric SU(2) Yang–Mills fields described on the 8th
dimensional superspace, the local gauge transformations constitute a group whose Lie algebra has
its coefficients belonging to the Weyl-spinorial Grassmann algebra.

We present here a Baker–Campbell–Hausdorff formula for the chiral SU(2) supergroup and
using this formula we give the finite form of each element of this group in terms of the local fields
entering in the infinitesimal real superscalar generator.

The natural generalization to superspace of a Lie algebra-valued real vector field

\[ \hat{A} = \hat{A}_\mu(x) \, dx^\mu = A^a_\mu \, X_a \, dx^\mu \quad (X_a^i = -X_i, \, A^a_\mu \, dx^\mu \, \text{real}) \]

is a Lie algebra-valued real supervector field

\[ \hat{\nu} = \hat{\nu}_A(z) \, \omega^A = \hat{\nu}_A^a(z) \, X_a \, \omega^A, \]

with \( \hat{\nu}_A \) real (which means \( (\nu^a_A \, \omega^A) = \nu^a_A \, \omega^A \)) [1].

The transformation law for this real supervector field is given by

\[ \hat{\nu}_A \rightarrow \nu(z) \hat{\nu}_A R^{-1}(z) - RD_A R^{-1}, \quad (1) \]

where

\[ D_A(\cdot) = [\partial_\mu / \partial_{(\cdot)}^\mu_A - i \partial_\mu(\cdot) \sigma_{\alpha \beta} \partial_{(\cdot)}^\delta / \partial_{(\cdot)}^A - i \partial_\mu(\cdot) \sigma_{\alpha \beta} \partial_{(\cdot)}^A] \]

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is the supersymmetric covariant derivative and $R(z)$ is the element of the supergroup we are interested in, generated by the infinitesimal real superscalars

$$\xi^a(z) = \xi^a(x) : R(z) = \exp \{ \xi^a(x) X_a \}.$$  

The field structure of $\xi^a(z)$ is given by

$$\xi^a(z) = \xi^a(x) + \xi^a_\alpha(x) \theta^\alpha - \xi^a_\beta(x) \bar{\theta}^\beta + \xi^a_\mu(x) \theta_+ - \xi^a_\nu(x) \bar{\theta}_- + \xi^a_\chi(x) \theta^\chi + \xi^a_\gamma(x) \bar{\theta}_\gamma + \xi^a_\delta(x) \theta^\delta + \xi^a_\epsilon(x) \bar{\theta}_\epsilon, \quad (2a)$$

\[ \xi^a, \xi^b, \xi^c \text{ reals}, \quad (2b) \]

$$\theta_+ = c_\alpha \theta^\alpha, \quad \theta_- = c_\beta \bar{\theta}^\beta \quad \theta_+ = \sigma_\mu \theta^\epsilon \bar{\theta}_\epsilon, \quad (2c)$$

In the following, we shall restrict our considerations to the super-subgroup constituted by elements $R$ generated by chiral superscalars

$$R = \exp \{ \xi^a(x) X_a + \xi^a_\alpha(x) \theta^\alpha + \xi^a_\beta(x) \bar{\theta}^\beta \}, \quad \xi^a = \xi^a X_a : \xi^a \text{ reals}. \quad (3a)$$

Our aim is to produce a finite expression for this element $R$ in the case $X_a^+ = X_a = (i/2) \sigma_a$ which are the anti-Hermitian generators of the $SU(2)$ Lie algebra.

In order to achieve this point, let us evaluate the group element $S$ defined by

$$S = \exp(Z) \cdot \xi^\prime \cdot \exp(-Z), \quad (3a)$$

where

$$Z = \xi_\alpha \theta^\alpha + \xi_+ \theta_+, \quad (3b)$$

$$\xi^\prime = (1 + c_\alpha \theta^\alpha + c_+ \theta_+) \xi. \quad (3c)$$

Throughout, we are using the abridged notation with Greek letters $\xi, \xi_\alpha, \xi_+$ to represent the Lie-valued objects $\xi^a X_a, \xi^a_\alpha X_a, \xi^a_\beta X_a, \xi^a_\gamma X_a$, respectively, and Lie scalars like $c_1, c_+$ are denoted with roman letters.

Since $[Z, [Z, Z, [Z, Z, Z]]] = 0$, $S$ can be written in the form

$$S = \xi^\prime + [Z, \xi^\prime] + \frac{1}{2!} [Z, [Z, [Z, \xi^\prime]]]$$

$$= \xi + [c_\alpha \xi + \xi_\alpha + c_+ \theta_+] \theta^\alpha +$$

$$+ [c_\alpha \xi + \xi_\alpha + 2^\epsilon e^{\alpha\beta} c_\mu \xi_\mu \xi_\alpha + 2^2 e^{\alpha\gamma} (\xi_\alpha \cdot \xi_\gamma) \xi - 2^2 e^{\alpha\gamma} (\xi \cdot \xi) \xi_\gamma] \bar{\theta}_+, \quad (4)$$