COMPLETELY INTEGRABLE THEORIES AS THEORIES OF SPONTANEOUS BREAKDOWN

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ABSTRACT. It is shown that the field theory described by a completely integrable equation (in particular, the theory of any free field) is the theory of spontaneous breakdown of symmetry under certain infinite dynamical group, the corresponding field being the Goldstone field by which this breakdown is accompanied.

1. The completely integrable equations currently attract considerable attention. They can be investigated in detail and have a number of interesting properties — infinite sets of integrals of motion, solitons, etc. (see the reviews [1, 2]). These specific properties seem to be associated with special symmetry properties of completely integrable equations. Particularly, the group of transformations transforming any solution of a certain completely integrable equation into any other solution of the same equation (dynamical group $\mathcal{D}$) is, as shown in [3, 4], an infinite group and contains an infinite group of symmetry of the type $G_{n=\infty}$, as well as an infinite Abelian group $B$ of Backlund transformations as subgroups.

In the present paper, we show that a theory described by some completely integrable equation is of the Nambu–Goldstone type, namely that it can be obtained by a nonlinear (Nambu–Goldstone) realization of the symmetry under the dynamical group $\mathcal{D}$, with the subgroup $G_{n=\infty}$ as a vacuum stability subgroup.

2. Let us first consider the simplest integrable equations — those linear over the field $\psi(x)$:

$$\mathcal{F}(\partial/\partial x) \psi(x) = 0,$$

where $x = (x_{\mu})$ are the space-time coordinates ($\mu = 0, 1, ..., N - 1$), $\mathcal{F}(\partial/\partial x)$ is an arbitrary differential operator. The space-time dimensionality $N$ and the number of field components (in particular, the spin) are also arbitrary.

The dynamical group $\mathcal{D}$ of field equation (1) (see for more detail [4]) has a simple structure. Let us consider, for simplicity, infinitesimal transformations $\psi \rightarrow \psi' = \psi + \delta \psi$ belonging to a dynamical group. Then, by definition of a dynamical group

$$\mathcal{F}(\partial/\partial x) \delta \psi(x) = 0.$$
Taking into account (1), one finds that $\delta \psi(x)$ is of a form:

$$\delta \psi(x) = \sum_{\alpha} a_{\alpha} D_{\alpha}(x) \omega(x)$$  \hspace{1cm} (2)

where $a_{\alpha}$ are the transformation parameters (arbitrary numbers), $D_{\alpha}(x)$ are differential operators commuting with $\mathcal{F}(\partial/\partial x)$, and $\omega(x)$ is an arbitrary solution of eqn. (1). The transformation of type (2) can be represented in the form of a combination of transformations of two types:

$$\delta \psi(x) = \sum_{\alpha} a_{\alpha} D_{\alpha}(x) \psi(x),$$  \hspace{1cm} (3)

$$\delta \psi(x) = \omega(x).$$  \hspace{1cm} (4)

Let us emphasize that the transformations of type (3) are homogeneous over the field $\psi(x)$, while the transformations (4) are inhomogeneous. That difference leads to the deep differences between the roles of transformations of types (3) and (4).

The transformations (3) are the transformations of the symmetry group of eqn. (1) in the infinitesimal form. In [3], it has been shown that the symmetry group of eqn. (1) is infinite and belongs to the type $G_{n=\infty}$. Indeed, if eqn. (1) is invariant under the $n$-parametric group $G_n$ of transformations

$$\delta \psi(x) = \sum_{i=1}^{n} b_i \Gamma_{i}(x) \psi(x),$$

i.e.

$$[\Gamma_{i}(x), \mathcal{F}(\partial/\partial x)] = 0, \hspace{1cm} (i = 1, ..., n),$$

then it is not difficult to show that eqn. (1) is invariant under the transformations

$$\delta \psi(x) = \sum_{i=1}^{n} b_i (\Gamma_{1}, ..., \Gamma_{n}) \Gamma_{i}(x) \psi(x),$$

where $b_i (\Gamma_{1}, ..., \Gamma_{n})$ are arbitrary functions of the infinitesimal generators $\Gamma_{1}(x), ..., \Gamma_{n}(x)$ of the symmetry group $G_n$. Thus, the total symmetry group of eqn. (1) is infinite and we denote it by $G_{n=\infty}$ to emphasize that the structure of the infinite group $G_{n=\infty}$ is completely determined by the structure of the $n$-parametric group $G_n$ [3]. The transformations (4) also form an infinite group, since $\omega(x)$ is an arbitrary solution of eqn. (1) and the number of independent solutions of such an equation is infinite. An arbitrary element of this group is of the form

$$g = \exp \left(-i \int d^{N-1}x \omega(x) \pi(x) \right),$$

where $\pi(x)$ are the canonical variables conjugate to the field $\psi(x)$. The group of transformations of type (4) is an infinite Abelian group [4] and can be identified as a group $B$ of Bäcklund trans-