On a property of conformal spacetimes

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There are spacetimes which do not possess a time-like isometry. In this paper we prove that if the metrics of these spacetimes are a particular generalisation of Carter’s Circular metrics [1] (which admit of a conformal time-like Killing vector [2]) the conformal time-like Killing vector depends only on time.

Time-like Killing vectors are required to define the energy of a test particle [3]

\[ E = k \cdot p \]  \hspace{1cm} (1)

where \( k \) is the time-like Killing vector and \( p \) the 4-momentum of the test particle. Realistically, of course, we do not expect a time-like Killing vector to exist in a cosmological context. Even in the simple Friedmann models [3] there exists only a conformal time-like Killing vector. In this case we can define a conformal analogue of energy

\[ \tilde{E} = k \cdot p \]  \hspace{1cm} (2)

\( k \) being a conformal time-like Killing vector i.e. a vector with respect to which the Lie derivative of the conformally transformed metric \( \tilde{g} = \Omega^2 g \) is zero. It satisfies the conformal Killing equation

\[ \tilde{g}_{ab,c} \tilde{k}^c + \tilde{g}_{ac} k^c_{,b} + \tilde{g}_{bc} k^c_{,a} = 0. \]  \hspace{1cm} (3)

In developing the “no-hair theorem” [3] Carter defined what he called “Circular metrics” [1]. These are metrics having at least one time-like and one space-like isometry. Choosing the isometry directions to give the \( x^0 \) and \( x^3 \) coordinates the metric is circular if it is expressible in the form

\[ g_{ab} = \begin{pmatrix} \frac{g_{rs}(x^q)}{0} & 0 & q, r, s = 1, 2 \\ 0 & \frac{g_{ab}(x^q)}{0} & \alpha, \beta = 3, 0 \end{pmatrix} \]  \hspace{1cm} (4)

To be able to define \( \psi N \) forces [4] and \( \psi N \) potentials [5] the circular metrics are extended to spacetimes with generalised circular metrics (GCMs). The GCMs included the following cases: (GCM1) when there are two space-like and one time-like Killing vector and \( q, r, s = 1 \) while \( \alpha, \beta = 2, 3, 0 \) in Eq. (4); (GCM2) when there is only one time-like Killing vector and \( q, r, s = 1, 2, 3 \) while \( \alpha, \beta = 0 \) in Eq. (4) and (CM) the Circular metric.

We need to generalise the GCMs to be able to extend the ideas of \( \psi N \) forces and potentials to spacetimes admitting only conformal isometries. Thus we define the conformal generalised circular metrics (CGCMs) by

\[ \tilde{g}_{ab} = \begin{pmatrix} \frac{\tilde{g}_{rs}(x^q)}{0} & 0 \\ 0 & \tilde{g}_{ab}(x^q) \end{pmatrix} \]  \hspace{1cm} (5)

with the corresponding requirements of the existence of Killing vectors and the range of the indices.
Now the time-like solution of Eq. (3) in a \textit{CGCM} has only the time component (by appropriate choice of the time direction),

\[ K^2 = f(t, x^i) \delta_0^0 \]  

(6)

where \( t \) is the time in these coordinates and \( i = 1, 2, 3 \). Now Eq. (3) gives

\[ \ddot{g}_{00} + 2 \dot{g}_{00} f, \dot{0} = 0 \]  

(7)

\[ \ddot{g}_{0i} + 2 \dot{g}_{0(i} f, \dot{0}) = 0 \]  

(8)

\[ \ddot{g}_{ij} + 2 \dot{g}_{0(i} f, \dot{j}) = 0. \]  

(9)

Equation (7), (8) and (9) can be solved to get

\[ g_{00} = A/f^2 \]  

(10)

\[ g_{0i} = 1/f (A a_i + B_i) \]  

(11)

\[ g_{ij} = 2 \{ A a_i a_j - \int a_i b_j dt - B_i a_j \} + C_{ij} \]  

(12)

where \( A, b_i, C_{ij} \) depend on space coordinates and

\[ \int (1/f)_i dt = a_i, \quad b_i = a_i,0. \]  

(13)

In the case of \textit{CGCM}, Eqs. (11) and (12) imply that

\[ B_i = 0, \quad a_i = 0, \quad \ddot{g}_{ij} = C_{ij}. \]  

(14)

Hence Eq. (13) implies that \( f_i = 0 \). Thus, from Eq. (6) a timelike solution of Eq. (3) (for a \textit{CGCM}) can depend only on time. Now Eq. (10) gives \( g_{00} = A/f^2 \) which is not in a \textit{GCM} form. We can rescale the time coordinate by defining a new "time"

\[ \tau = \int dt/f(t) \]  

(15)

so that

\[ g_{00} d\tau^2 = \ddot{g}_{00} dt^2 \]  

(16)

which gives

\[ g_{00} = A \]  

(17)

as is required for the \textit{CGCM}. Thus we state the following: Theorem:

A spacetime whose metrics are a conformal generalisation of Carter's Circular metrics admit of a conformal time-like Killing vector and the vector depends only on time.

References