EXTENSION OF MONOTONE NORM FROM A NORMED STRUCTURE TO ITS DEDEKIND AUGMENTATION

V. A. Solov'ev

In the present work monotone extension of the norm is considered from a KN-lineal to its K-augmentation (that is, to its Dedekind augmentation).† The definitions of basic concepts employed here can be found in [1] and [2].

Let X be a KN-lineal, and X its K-augmentation. In [1] a method was given for monotone extension of the norm from X to X, namely, for each x ∈ X one set

$$\|\tilde{x}\| = \inf_{x \in X; \tilde{x} \geq |x|} |x|.$$

Such extension will be called a natural extension. It can easily be shown that generally speaking other monotone extensions of the norm from X to X may exist, different from natural. However, it is not difficult to see that if \(||^*\) denotes any monotone extension of the norm \(||\) from X to X and \(\tilde{x} \in \tilde{X}\) then the following inequality holds:

$$\sup_{y \in X; \tilde{x} \leq |y|} |y| \leq \|\tilde{x}\| \leq \inf_{x \in X; \tilde{x} \geq |x|} |x|.$$

First, we shall outline a sufficiently general scheme for constructing such extensions.

Let X be a KN-lineal with the property that extension of regular functionals on X from X to X is in general not unique. Then there exists a positive functional on X with at least two positive extensions to \(\tilde{X}\). Indeed, let any positive functional on X have a unique positive extension. Then any regular functional on X will also have a unique regular extension to \(\tilde{X}\). In fact, let \(\tilde{f}_1\) and \(\tilde{f}_2\) be regular functionals on X such that \(\tilde{f}_1 (x) = \tilde{f}_2 (x) = f (x)\) for \(x \in X\); then \(\tilde{f}_1 - \tilde{f}_2\) is a regular extension of the null functional. On the other hand, it is known that \(\tilde{f}_1 - \tilde{f}_2 = \tilde{g}_1 - \tilde{g}_2\), where \(\tilde{g}_1\) and \(\tilde{g}_2\) are positive functionals on X; moreover, \(\tilde{g}_1 (x) = \tilde{g}_2 (x)\) for \(x \in X\); consequently, by our assumption since \(\tilde{g}_1, \tilde{g}_2 \geq 0\), one has \(\tilde{g}_1 = \tilde{g}_2\) and therefore \(\tilde{f}_1 = \tilde{f}_2\). From the outset we have considered such X for which extension of regular functionals is not unique.

Thus we take two positive functionals on X, \(\tilde{f}\) and \(\tilde{g}\) such that \(\tilde{f}(x) = \tilde{g}(x)\) for \(x \in X\), but \(\tilde{f} \neq \tilde{g}\); we introduce in X the new norm \(\|x\|_1 = \|x\| + \tilde{f} (|x|), x \in X\).

The latter has two different monotone extensions onto \(\tilde{X}\) (we denote by \(\|\|^{*}\) the natural norm extension):

$$\|\tilde{x}\|_1 = \|\tilde{x}\|^{*} + \tilde{f} (|\tilde{x}|) (\tilde{x} \in \tilde{X}).$$

$$\|\tilde{x}\|_2 = \|\tilde{x}\|^{*} + g (|\tilde{x}|) (\tilde{x} \in \tilde{X}).$$

It is not difficult to show that a monotone extension of norm from a KB-lineal X on its K-augmentation X is not generally unique either, not even in the case when the (b)-completeness of X is demanded with regard to the extended norm.

† By monotone norm extension we mean an extension which maintains monotonicity. As it is known, a norm on a K-lineal X is called monotone if \(|x| \leq |y|\) implies \(\|x\| \leq \|y\|\).

† For example, if one considers a K-lineal c and its K-augmentation m then it follows from Theorem 3 of [6] that a regular functional exists on c whose extension to m is not unique.

To this end we consider a KB-lineal c and its K-augmentation m. We introduced the following norm in m: if \( x = \{ \alpha_i \}_{i=1}^{\infty} \), we set
\[
\| x \|_1 = \sup_i |\alpha_i| + \varphi(|x|),
\]
where \( \varphi(x) \) is the Banach limit of the sequence \( \{ \alpha_i \}_{i=1}^{\infty} \); in addition, we introduce another norm in m: \( \| x \|_2 = \sup_i |\alpha_i| + \psi(|x|) \), where \( \psi(x) = \lim_{k \to \infty} \alpha_k \).

Obviously, on c one has \( \| x \|_1 = \| x \|_2 \), but on m they are, generally speaking, not identical. We note that \( \varphi(|x|) \) and \( \psi(|x|) \) are convex functionals which are \( (b) \)-continuous with respect to the usual norm in c and m: \( \| x \| = \sup_i |\alpha_i| \). It follows easily from convexity and \( (b) \)-continuity of \( \varphi(|x|) \) and \( \psi(|x|) \) that c and m are \( (b) \)-complete under both norms, \( \| \cdot \|_1 \), and \( \| \cdot \|_2 \). Thus, even if \( (b) \)-completeness is retained in \( X \), a monotone extension of norm is generally not unique.

Before discussing uniqueness conditions of monotone extensions of a norm from KN-lineal X to its K-augmentation \( \hat{X} \) we shall prove a theorem of extending \( (b) \)-linear functionals from X to \( \hat{X} \) in the case of the natural norm extension.

**THEOREM 1.** In the natural norm extension any positive extension from X to \( \hat{X} \) of a positive \( (b) \)-linear functional \( f \) is \( (b) \)-linear and maintains the norm \( \| f \| x = \| f \| \hat{x} \).

**Proof.** We shall show that
\[
\| f \| \hat{x} = \sup_{t < 1} \| f \| x \leq \infty.
\]
It follows from the definition of the natural norm extension from X to \( \hat{X} \) that for each \( \hat{x} \in \hat{X} \) with \( \| \hat{x} \| < 1 \) a \( x \in X \) exists such that \( x \geq x \) and \( \| x \| < 1 \). As \( f \to 0 \) for any \( x \in X \) with \( \| x \| < 1 \), therefore an \( \hat{x} \in X \), exists such that \( \| x \| < 1, x \geq \hat{x} \) and \( f(x) \leq f(\hat{x}) \); consequently,
\[
\sup_{t < 1} \| f \| x \leq \sup_{t < 1} \| f \| \hat{x} \leq \| f \| \hat{x}.
\]
On the other hand, the reverse inequality is obvious, and therefore \( \| f \| x = \| f \| \hat{x} \).

**COROLLARY.** If one denotes the space associated with X by \( \bar{X} \), and \( (b) \)-adjoint by \( X* \), then \( \bar{X} = X* \) implies that \( \hat{X} = X* \).

Note. Theorem 1 does not hold in general for monotone extended norms that are not natural, that is, not every positive extension of a positive and \( (b) \) linear functional on X is \( (b) \)-linear.

To prove this we shall consider the following example. For any sequence \( x = \{ a_i \}_{i=1}^{\infty} \) we denote by \( \psi(x) \) its Césaro limit, \( \psi(x) = \lim_{n \to \infty} \frac{a_1 + \ldots + a_n}{n} \). It is know that for \( x \in X \) one has \( \psi(x) = \lim_{k \to \infty} \alpha_k \).

We consider \( \psi(x) \) on the set \( \varepsilon \cap m \), where \( \varepsilon \) denotes the totality of sequences which are Césaro convergent; obviously, \( \psi(x) \) is a positive functional; the set \( \varepsilon \cap m \) is linear and it majorizes m; consequently, \( \psi \) admits a positive extension from \( \varepsilon \cap m \) to the entire m (we denote this extension also by \( \psi \)).

We define now two norms on m:
\[
\| x \|_1 = \sup_k \frac{|a_k|}{k} + \psi(|x|), \quad \| x \|_2 = \sup_k \frac{|a_k|}{k} + \xi(|x|),
\]
where \( \xi(|x|) = \lim_{k \to \infty} \frac{1}{k} \).

We consider a sequence \( \{ x_n \} \in m \), generated in the following manner \( \hat{x}_n = \{ a_k^{(n)} \}_{k=1}^{\infty} \) \( (n = 1, 2, \ldots) \), where
\[
a_k^{(n)} = \begin{cases} 1 & \text{for } k = 1^2, \text{ if } i^2 \geq n, \\ 0 & \text{for other } k. \end{cases}
\]

Obviously, \( \sup_k \{ |a_k^{(n)}|/k \} \to 0 \). It can easily be verified that \( \psi(\hat{x}_n) = 0 \) for all n since
\[
0 \leq \frac{a_1^{(n)} + a_2^{(n)} + \ldots + a_k^{(n)}}{k} \leq \frac{\xi}{k}.
\]

For definition of a convex functional see [3] (p. 233). Here we consider convex functionals which assume finite values only.