ON THE THEORY OF DEGENERATE ELLIPTIC EQUATIONS

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For ordinary differential equations there is a well developed theory of the behavior of solutions in a neighborhood of points where the order of the equation is degenerate (see [1]-[3]). Very thorough investigations have been made concerning so-called regular singular points of integrals of ordinary differential equations (see [3]). For degenerate elliptic equations, although many problems have been studied in connection with the behavior of solutions in a neighborhood of manifolds where the equations are degenerate (see [4]-[6]), the theory of the behavior of solutions of these equations in a neighborhood of such manifolds of degeneracy is still not nearly as complete as the theory of the behavior of solutions of ordinary differential equations in a neighborhood of points of degeneracy.

The works [7]-[9] were devoted to a systematic generalization of the theory of singular points of integrals of ordinary differential equations to the behavior of solutions of elliptic equations in a neighborhood of manifolds where the order is degenerate. In those works the techniques of ordinary differential equations were modified in order to investigate degenerate elliptic equations in a neighborhood of points of a manifold of degeneracy of the equation.

In [10] integral representations of functions of several complex variables were applied to the study of the behavior of solutions of degenerate elliptic equations in a neighborhood of a manifold of degeneracy. This method is suitable for studying equations of the special form

$$z^2(u_{zz} + L(z)u) + a(z)u_{z} + b(z)u = 0,$$

where

$$L = \sum_{i,j=1}^{m} a_{ij}(X) \frac{\partial^2}{\partial x_i \partial x_j}, \quad X = (x_1, \ldots, x_m),$$

is an operator with analytic coefficients which is real for real values of the independent variables, and for these values the operator $L$ is elliptic. If for a domain $D$ which is contained in the domain of analyticity of the coefficients of $L$ there exists a kernel $K(X, X')$ of an integral representation of functions $f(X)$ which are holomorphic in $D$ (see [11]), satisfying the conditions:

a) $K(X, X')$ is holomorphic in the variables $X = (x_1, \ldots, x_m)$ and $X' = (x_1', \ldots, x_m')$ and for all $X \in D$;

b) $L \gamma_k(K(X, X')) = \gamma_k K(X, X) \overline{M(X, X') \gamma_k}, \quad k = 0, 1, \ldots,$

where the $\gamma_k$ are complex numbers which satisfy the inequalities

$$|\gamma_k| \leq C \cdot (kt)^{\delta}, \quad k = 0, 1, \ldots,$$

and $M(X, X')$ is a function which is holomorphic in all variables;

then in order to study the behavior of solutions of equation (1) in a neighborhood of points of the hyperplane $z = 0$ it suffices to study solutions of the form

$$v = K(X, X) \sum_{n=0}^{\infty} \psi_n(z) [M(X, X)]^{\gamma_n},$$

$$w = K(X, X) \sum_{n=0}^{\infty} \psi_n(z) [M(X, X)]^{\gamma_n},$$

where the functions $\varphi_n$ and $\psi_n$ satisfy the recursion relation
\[ z^n \varphi_n'' + a(z) z \varphi_n' + b(z) \varphi_n = -\gamma_n \omega_{n-1}, \quad n=0, 1, \ldots, \]
and the initial conditions for $z = \xi \neq 0$.
\[ \varphi_0(\xi) = 1, \quad \varphi_0'(\xi) = 0, \quad \varphi_n(\xi) = \varphi_n'(\xi) = 0, \quad n=1, 2, \ldots. \]
\[ \psi_0(\xi) = 0, \quad \psi_0'(\xi) = 1, \quad \psi_n(\xi) = \psi_n'(\xi) = 0, \quad n=1, 2, \ldots. \]

From condition (2) it follows that
\[ L^{k+1}(K(X, \overline{X})) = \gamma_{k+1} K(X, \overline{X}) [M(X, \overline{X})]^{2k+2} \]
\[ = \gamma_k L(K(X, \overline{X}) [M(X, \overline{X})]^n). \quad (4) \]

Then for $k = 0$ we have
\[ L(K(X, \overline{X})) = \gamma_0 K(X, \overline{X}) [M(X, \overline{X})]^3. \quad (5) \]

From (4) and (5) we obtain
\[ \frac{\gamma_{k+1}}{\gamma_k} M^4 K = 2k M \sum_{i,j=1}^m a_{ij}(X) \left( \frac{\partial K}{\partial x_i} \frac{\partial M}{\partial x_j} + \frac{\partial K}{\partial x_j} \frac{\partial M}{\partial x_i} \right) + 2k(2k-1) K \sum_{i,j=1}^m a_{ij}(X) \frac{\partial M}{\partial x_i} \frac{\partial M}{\partial x_j} + \gamma_1 M^4 K + 2k M K L(M). \quad (6) \]

From this equation it follows that there exists a holomorphic function $H(X, \overline{X})$ such that
\[ \sum_{i,j=1}^m a_{ij}(X) \left( \frac{\partial K}{\partial x_i} \frac{\partial M}{\partial x_j} + \frac{\partial K}{\partial x_j} \frac{\partial M}{\partial x_i} \right) = H(X, \overline{X}) K(X, \overline{X}). \]

Now we see that
\[ \frac{\gamma_{k+1}}{2k \gamma_k} = (2k - 1) \sum_{i,j=1}^m a_{ij}(X) \frac{\partial M}{\partial x_i} \frac{\partial M}{\partial x_j} + MH + ML(M). \quad (7) \]

If we set $k = 1$ and $k = 2$ in (7) we obtain
\[ \sum_{i,j=1}^m a_{ij}(X) \frac{\partial M}{\partial x_i} \frac{\partial M}{\partial x_j} = \frac{\beta - \alpha}{2} M^4, \quad (8) \]
\[ \sum_{i,j=1}^m a_{ij}(X) \left( \frac{\partial K}{\partial x_i} \frac{\partial M}{\partial x_j} + \frac{\partial K}{\partial x_j} \frac{\partial M}{\partial x_i} \right) + KL(M) = \frac{3\alpha - \beta}{2} K M^3, \]
where
\[ \alpha = \frac{\gamma_2 - \gamma_1^2}{2\gamma_1}, \quad \beta = \frac{\gamma_2 - \gamma_2^2}{4\gamma_2}. \]

By virtue of (8) we see that (7) implies that
\[ \gamma_{k+1} / \gamma_k = \gamma_1 + 2k^2(\beta - \alpha) + 2k(2\alpha - \beta), \]
from which we obtain
\[ \gamma_k = 2^{k-1}(\beta - \alpha)^{k-1} \frac{\Gamma(k + s_1) \Gamma(k + s_2)}{\Gamma(s_2 - 1) \Gamma(s_2 - 4)}, \quad (9) \]
where
\[ s_1 = (\beta - \alpha)^{-1}(2\alpha - \beta - \gamma(2\alpha - \beta)^2 - 4\gamma_1 (\beta - \alpha)), \]
\[ s_2 = (\beta - \alpha)^{-1}(2\alpha - \beta + \gamma(2\alpha - \beta)^2 - 4\gamma_1 (\beta - \alpha)). \]

We can consider a more general case. Let $K(X, \overline{X})$ be an arbitrary kernel of an integral representation of analytic functions of several complex variables such that the kernel is holomorphic in $X$ and $\overline{X}$. We consider the series