A CONDITION OF EXISTENCE OF THE CURVATURE TENSOR

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INTRODUCTION

In this paper we introduce a counterpart of the curvature tensor for a differentiable manifold of class $C^2$ with linear connectivity that has continuous coefficients $\Gamma^k_{ij}$ and satisfies certain additional conditions. In general it is not assumed that the $\Gamma^k_{ij}$ are differentiable.

In a $C^2$-manifold with linear connectivity that has continuously differentiable coefficients $\Gamma^k_{ij}$, the curvature tensor field $R$ assigns to each ordered triple of continuous vector fields $(X, Y, Z)$ a continuous vector field $R(X, Y)Z$. For a vector triple $(u, v, w)$ at a given point, $R(u, v)w$ indicates the deviation (from its original value) of a vector $w$ that has performed a parallel translation along an infinitely small closed contour that lies in a two-dimensional oriented direction specified by the vector pair $(u, v)$.

Thus there exists a geometrical construction that permits the calculation of the curvature tensor.

We shall assume that in the case of continuous $\Gamma^k_{ij}$, this construction assigns a continuous vector field $R(X, Y)Z$ to each triple of continuous vector fields $(X, Y, Z)$. We shall prove here that under this assumption, $R$ is a tensor.

1. FORMULATION OF RESULTS

1. Geometrical Meaning of Curvature Tensor of Linear Connectivity. At first let us recall the geometrical meaning of a curvature tensor $R$ of linear connectivity with continuously differentiable coefficients $\Gamma^k_{ij}$ in a manifold $M^d$ of class $C^2$.

By $E^2$ we shall denote a two-dimensional Euclidean vector space with a selected orthonormalized base $\{e_1, e_2\}$ that specifies the orientation of $E^2$.

For a given ordered triple $(u, v, w)$ of vectors tangent to $M^d$ at the point $p$, it is possible to obtain the vector $R(u, v)w$ as follows.

1) We assign a mapping $f$ of the class $C^2$ (belonging to $E^2$) into $M^d$ such that

a) $f(0) = p$,

b) the vectors $df(e_1)$ and $df(e_2)$ tangent to the paths $f(t e_1)$ and $f(t e_2)$, $0 \leq t \leq 1$, coincide for $t = 0$ with $u$ and $v$, respectively.
2) We assign a family \( \{\tau(s)\} \) \((0 < s < \delta, \delta > 0)\) of closed piecewise-smooth non-self-intersecting positively oriented paths in \( \mathbb{E}^2 \) that contract to \( O \) when \( s \to 0 \), and that bound in \( \mathbb{E}^2 \) regions with areas \( \sigma(s) \) that are second-order infinitesimals with respect to the lengths \( ||\tau(s)|| \) of the paths \( \tau(s) \).

3) We assign a family \( \{w(s)\} \) of vectors that are tangent at the end of the path \( f(\tau(s)) \) and such that \( w(s) \to w \) for \( s \to 0 \).

4) By \( w_1(s) \) we denote a vector at the beginning of the path \( f(\tau(s)) \) that goes over into the vector \( w(s) \) by a parallel translation along the path \( f(\tau(s)) \).

Under the conditions 1)-4) there exists a limit
\[
\lim_{s \to 0} \frac{[w_1(s) - w(s)]}{\sigma(s)} = R(u, v)w
\]
(see [1]).

2. Formulation of Tensor Theorem. Let \( M^d \) be a manifold of class \( C^2 \) with linear connectivity that has continuous coefficients \( f^k_i \) in the maps of the atlas of a differentiable structure \( M^d \).

Let us assume that the following conditions A and B are satisfied.

A) For any \( p \) belonging to \( M^d \), and \( u, v, w \) belonging to the space \( M^d_p \) tangent at the point \( p \), and any \( f, \{\tau(s)\}, \{w(s)\}, \{w_1(s)\} \) that satisfy the conditions 1)-4) of Sec. 1, the limit \( \lim_{s \to 0} \frac{[w_1(s) - w(s)]}{\sigma(s)} \) exists and depends only on \( (u, v, w) \).

By definition let us write
\[
R(u, v)w = \lim_{s \to 0} \frac{[w_1(s) - w(s)]}{\sigma(s)}.
\]

(2)

B) For any continuous vector fields \( X, Y, Z \) on \( M^d \), the vector field \( R(X, Y)Z \) defined by the formula
\[
(R(X, Y)Z)(p) = \langle X(p), Y(p) \rangle Z(p),
\]
is continuous.

We shall prove the following.

THEOREM. If the conditions A and B are satisfied, \( R \) will be a tensor.

2. Principal Construction in the Proof of the Theorem

In this section we shall present the principal construction. Thus the proof of the theorem reduces to auxiliary assertions that will be proved in Sec. 3.

1. Preliminary Remarks and Lemmas. Remark 1. First of all let us note that for fixed \( u \) and \( v \) in \( M^d_p \), the mapping that assigns a vector \( R(u, v)w \) to a vector \( w \) in \( M^d_p \) will be linear. This is a direct consequence of formula (2). Let us denote this linear mapping by \( R(u, v) \). For proving the theorem, it suffices to prove the bilinear dependence of \( R(u, v) \) on \( u \) and \( v \).

LEMMA 1. For any \( u \) and \( v \) in \( M^d_p \) and any real number \( \alpha \) we have
\[
R(\alpha u, v) = \alpha R(u, v),
\]
\[
R(u, \alpha v) = -R(v, u).
\]

(3)

(4)

Proof. Let us prove (4). For \( (u, v, w) \) let us specify quantities \( f, \{\tau(s)\}, \{w(s)\}, \{w_1(s)\} \) that satisfy the conditions 1)-4). By \( h \) we denote a linear mapping of \( \mathbb{E}^2 \) into itself such that
\[
h(e_1) = e_2, \quad h(e_2) = e_1.
\]

Let us define \( \tilde{f}, \{\tilde{\tau}(s)\}, \{\tilde{w}(s)\}, \{\tilde{w}_1(s)\} \) by the relations
\[
\tilde{f} = fh, \quad \tilde{\tau}(s) = h^{-1}(\tau(s))^{-1}, \quad \tilde{w}(s) = w_1(s), \quad \tilde{w}_1(s) = w(s).
\]

Here we denoted by \( (\tau(s))^{-1} \) the inverse of the path \( \tau(s) \).

Then \( \tilde{f}, \{\tilde{\tau}(s)\}, \{\tilde{w}(s)\}, \{\tilde{w}_1(s)\} \) will satisfy the conditions 1)-4) for the triple \( (v, u, w) \).

Hence we obtain the formula (4). Formula (3) can be proved in a similar way. This completes the proof of the lemma.