FINDING THE FORM OF THE SURFACE OF A LIQUID IN A CONICAL CONTAINER FOR A GIVEN VOLUME OF THE LIQUID. I

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Let \( \Omega \) be a bounded domain in \( n \)-dimensional Euclidean space \( E^n; \Gamma = \partial \Omega \) be the boundary of the domain \( \Omega \) of class \( C^2 \). In this paper we consider the following problem. For a given number \( \beta \) it is required to find a number \( \lambda \) and a function \( u \in C^2(\Omega) \), which satisfy the function
\[
Lu = \sum_{\alpha=1}^{n} \frac{d}{dy^\alpha} \left[ \alpha^{-1} \Phi_i(u, \alpha - au) \right] + n \alpha^{-1} \Phi_{n+1}(u, -au) = \psi(y, u, \lambda), \quad y \in \Omega,
\]
the boundary condition
\[
L_{\Gamma}u = \sum_{i=1}^{n} \Phi_i(u, -au) \gamma_i = \chi(y, u), \quad y \in \Gamma,
\]
and the relation
\[
V(u) = \int_{\Omega} \varphi(y, u) \, dy = \beta.
\]

Here \( \gamma = (\gamma_1, \ldots, \gamma_n) \) is the unit normal to \( \Gamma \), outward in relation to \( \Omega \); \( u = (u_1, \ldots, u_n) \); \( \psi, \chi, \varphi, a \) are given functions, while \( a \) is continuously differentiable in \( \Omega \) and bounded below and above by positive constants \( \rho_1 \) and \( \rho_2 \); \( \Phi \) is a positive homogeneous function of \( q \in E^{n+1} \) of the first degree;
\[
\Phi_i(q) = \frac{\partial \Phi(q)}{\partial q_i}, \quad \Phi_{n+1}(q) = \frac{\partial \Phi(q)}{\partial q_{n+1}}, \quad i, j = 1, \ldots, n+1.
\]

Along with the problems posed, we shall consider the problem (I), (II), for fixed value of the parameter \( \lambda \). Let us agree to denote this problem by \((I_\lambda), (II_\lambda)\).

In the present paper we introduce the concept of barrier functions for the solution of the problem \((I_\lambda), (II_\lambda)\) and, under the assumption of the existence of these functions, we prove the classical solvability of the problem \((I_\lambda), (II_\lambda)\), and we also investigate the solvability of the basic problem (I), (II), (III).

For
\[
\Phi(q) = q; \quad \psi(y, t, \lambda) = k \frac{4 + y^2}{4 + y^2} \alpha \tau^2 - \lambda \tau, \quad k = \text{const} > 0,
\]
\[
a(y) = \frac{4 + y^2}{4 + y^2}, \quad y^2 = \sum_{i=1}^{n} y_i^2; \quad \varphi(y, t) = \frac{1}{a+1} \alpha \tau^{a+1}; \quad \Omega = \{y \in E^n : |y| < 2\}
\]
the problem (I), (II), (III) coincides with the problem of determining the form of the surface of a liquid in a conical container for a given volume of liquid (for more details on this, cf. [5]). For this concrete physical problem, in the second part of the paper, which was deposited in VINITI in 1981, barrier functions are constructed.

It is assumed that
\[
\Phi \in C^1(E^{n+1} \setminus \{0\})
\]
and there exist positive constants \( \nu_i \) and \( \mu_i \) (\( i = 0, 1, 2 \)) such that for any \( q \in E^{n+1} \) one has
\[
\nu_0 |q| < \Phi(q) < \mu_0 |q|.
\]
\[
v_1 \leq \left( \sum_{i=1}^{n+1} \Phi_{ij}(q_i) \right)^{1/2} \leq \xi_{ij},
\]

\[
\frac{\kappa}{q^{2}} \xi'' \leq \sum_{i,j=1}^{n+1} \Phi_{ij}(q) \xi_i \xi_j \leq \frac{\kappa}{q^{2}} \xi'', \quad \forall \xi \in \mathbb{E}^{n+1},
\]

where \( \xi'_i = \xi_i - \left( \sum_{i=1}^{n+1} \xi_i q_i \right) \frac{q_i}{q^2} \). We note that by virtue of the homogeneity of the function \( \Phi \)

\[
\sum_{i=1}^{n+1} \Phi_{ij}(q_i) q_i = 0, \quad \text{where} \quad j = 1, \ldots, n+1.
\]

First we shall consider the problem \((I, \Lambda), (II, \Lambda)\). A priori estimates of solutions of this problem will be established with the help of a comparison theorem. Before formulating this theorem, we give the definition of the regular conormal derivative corresponding to the operator \( L \).

Let \( \Omega \) be some bounded domain in \( \mathbb{E}^n \) with boundary \( \Gamma \) of class \( C^2 \). For sufficiently small \( \delta > 0 \) we consider the family of domains

\[
\Omega(\delta) = \{ y \in \Omega : \text{dist}(y, \Gamma) > \delta \}
\]

with boundary \( \Gamma(\delta) \) and by \( \gamma^\delta = (\gamma^\delta_1, \ldots, \gamma^\delta_n) \) we denote the unit normal vector to \( \Gamma(\delta) \), exterior in relation to \( \Omega(\delta) \).

**Definition 1.** The function \( v \in C^2(\Omega) \cap C(\overline{\Omega}) \) has on \( \Gamma \) a regular conormal derivative corresponding to the operator \( L \) if the expression

\[
L_{\Gamma}(\omega) = \sum_{i=1}^{n+1} \Phi_i (v_{\gamma_i} - av) \gamma_i^\delta,
\]

calculated at the point \( y_0 = y - \delta \gamma(y), \quad y \in \Gamma \), as \( \delta \to 0 \), tends to a limit uniformly with respect to \( y \in \Gamma(\delta) \). This limit will be called the regular conormal derivative of \( v \) on \( \Gamma \) and denoted by \( L_{\Gamma} v \).

**Theorem 1.** Let \( \Omega_0 \) be a domain with boundary \( \Gamma_0 \) of class \( C^2 \), having nonempty intersection with \( \Omega \). Let us assume that \( v \) and \( w \) are positive functions such that \( w \in C^2(\overline{\Omega}), \ v \in C^2(\Omega_0) \cap C(\overline{\Omega}_0) \) and on \( \Gamma_0 \) there exists the regular conormal derivative of the function \( v \). Then if

\[
L v < L w, \quad y \in (\Omega \cap \Omega_0),
\]

\[
L_{\Gamma_0} v \geq L_{\Gamma_0} w, \quad y \in (\Gamma_0 \cap \overline{\Omega}),
\]

\[
L v \geq L w, \quad y \in (\Gamma \cap \overline{\Omega}_0),
\]

then for all \( y \in \partial \Omega_0 \) one has \( v(y) \geq w(y) \).

**Proof.** We multiply the difference \( L v - L w \) by the function

\[
\eta = \begin{cases} 
\varepsilon_\beta - \varepsilon_\alpha, & \text{if } \varepsilon_\beta < \varepsilon_\alpha \\
0, & \text{if } \varepsilon_\beta \geq \varepsilon_\alpha 
\end{cases}
\]

and we integrate over the domain \( \Omega \cap \Omega_0(\delta) \). Further, applying integration by parts, we arrive at the relation

\[
\int_{(\Omega \cap \Omega_0(\delta))^-} n_{\beta} a^{-1} \left\{ \sum_{i=1}^{n+1} \Phi_i (v_{\gamma_i} - av) - \Phi_i (w_{\gamma_i} - aw) \right\} (v_{\gamma_i} a^{-1} - w_{\gamma_i} a^{-1}) + \\
+ \left[ \Phi_{n+1} (v_{\gamma_{n+1}} - av) - \Phi_{n+1} (w_{\gamma_{n+1}} - aw) \right] [(a^{-1} v - aw) a^{-1} - (aw) a^{-1}] \right\} d\gamma \\
+ \int_{(\Omega \cap \Omega_0(\delta))^-} (L v - L w) (\varepsilon_\beta - \varepsilon_\alpha) d\gamma = \\
\int_{(\Gamma \cap \partial \Omega_0(\delta))^-} a^{-1} (L_{\Gamma_0} v - L_{\Gamma_0} w) (\varepsilon_\beta - \varepsilon_\alpha) d\Gamma \\
+ \int_{(\Gamma_0 \cap \partial \Omega_0(\delta))^-} a^{-1} (L_{\Gamma_0} v - L_{\Gamma_0} w) (\varepsilon_\beta - \varepsilon_\alpha) d\Gamma.
\]