SINGULAR INTEGRAL OPERATORS IN WEIGHTED $L_2$ SPACES

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In this paper we consider singular integral operators (s.i.o.) with the symbol $\Phi(x, \xi)$, depending on a pole, in the space $L_2$ with the weight $|x|^\alpha = \rho^\alpha$.

1. Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space and let $S$ be its unit sphere. Let $L_{2,\alpha}$ be the space of functions defined on $\mathbb{R}^n$ for which we have the finite norm

$$
\|u\|_{L_{2,\alpha}} = \|\rho^{\alpha}u\|_2 < \infty.
$$

Let $\mathcal{D}$ be the space of infinitely differentiable functions with a compact support and let $\mathcal{D}_0$ be the set of functions $u \in \mathcal{D}$ for which $0 \notin \text{supp} u$. Let $Q_s$ ($s \geq 0$ is an integer) be the set of functions $u \in \mathcal{D}$ satisfying the condition $\int_{\mathbb{R}^n} x^\omega u(x) dx = 0$, where $\omega$ is a multi-index of order $0 \leq |\omega| \leq s$.

$Q_{-s}$ is the set of functions $u \in \mathcal{D}_0$ satisfying the condition

$$
\int_0^1 \rho^{-s} u(\rho, \theta) d\rho = 0, \quad q = 1, 2, \ldots, s, \quad 0 \leq \theta < \pi.
$$

$Q_{-s}$, $Q_{s}$, $Q_{-s}$ are dense in the spaces $L_{2,\alpha}$ (see [1, 2]). $W^s_2(\mathbb{R}^n)$ is the Sobolev-Slobodetskii space. By introducing a local system of coordinates on the unit sphere, one defines the spaces $W^s_2(S)$ [3, 4]. Let $\delta$ be the Beltrami operator on the sphere, let $E$ be the identity operator and let $\alpha$ be a real number. As shown in [3, 5], the norm $\| (E - \delta)^{\alpha/2} f \|_{L_2(S)}$ is equivalent to the norm $\| f \|_{W^s_2(S)}$. $L_\infty W^s_2(S)$ is the space of functions $f(x, \theta)$ which belong for almost all $x$ to the space $W^s_2(X)$ with respect to the variable $\theta$, and

$$
\| f \|_{L_\infty W^s_2(S)} = \text{vrai sup} \| f(x, \cdot) \|_{W^s_2(S)}.
$$

Following [5], to each function $f(\theta) \in L_2(S)$ we associate the series $f(\theta) = \sum_{k=0}^{\infty} \sum_{m=-\infty}^{\infty} a_{km} Y_{km}(\theta)$, where $Y_{km}(\theta)$ is the normed spherical function of homogeneity order $k$, $0 \leq k \leq \infty$, $m_k \sim k^{n-2}$ for $k \to \infty$.

It has been proved in [5] and [3] that the condition $f(x, \theta) \in L_\infty W^s_2(S)$ is equivalent to the boundedness of the series

$$
|a_{\alpha}(x)|^2 + \sum_{k=0}^{\infty} \sum_{m=-k}^{m_k} k^{2|\alpha|} |a_{km}(x)|^2
$$

uniformly with respect to $x$.

2. We denote by $A_{km}$ the s.i.o. with the symbol $Y_{km}$

$$
A_{km}u = F^{-1}\tilde{Y}_{km} \tilde{F}u,
$$

where $F$ is the Fourier transform and $F^{-1}$ is its inverse.

**Lemma 1.** Let $\alpha \geq 0$ be a real number. We have the inequality

$$
\sum_{m=0}^{m_k} \| (E - \delta)^{\alpha/2} Y_{km} f \|_{L_2(S)} \leq c k^{\alpha-2+\alpha} \| (E - \delta)^{\alpha/2} f \|_{L_2(S)}.
$$


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Proof. The proof of Lemma 1 makes use of the following estimate for spherical functions
\[ \sum_{m=0}^{\infty} |D_m Y_{km}|^2 \leq C k^{n-2\alpha} |\alpha|, \]
which improves the similar estimate given in [5]. For the derivation of the estimate (1) one makes use of the method of induction and of the addition theorem for Legendre polynomials.

For integer \( \alpha \), Lemma 1 follows immediately from (1).

For noninteger \( \alpha \) the proof reduces with the aid of a local system of coordinates on the sphere \( S \) and of a diffeomorphic mapping onto \( \mathbb{R}^{n-1} \) to the estimate of fractional derivatives of the function \( Y_{km} \).

Lemma 2. Let \( \alpha \) be a real number, \( |\alpha| \neq n/2 + r, r = 0, 1, \ldots \). Then, for each function \( u \in Q_S (Q_{-S} \text{ for } \alpha < 0), s = [\alpha] - n/2 \), we have the inequality
\[ \sum_{m=0}^{\infty} \|A_{km} u\|_{L^2}^2 \leq C k^{n-2\alpha s} \|u\|_{L^2}^2. \]

For \( |\alpha| < n/2 \), the membership of \( u \) in the set \( Q_S (Q_{-S} \) means that \( u \in \mathcal{D}(\mathcal{D}_0); \) \( [\cdot] \) denotes the integer part of a number.

Proof. The proof of Lemma 2 is carried out for \( \alpha > 0 \) and \( \alpha < 0 \). For \( \alpha > 0 \) one makes use of the following inequality [6]
\[ c_1 \|(-\Delta)^{\alpha/2} u\|_{L^2} \leq \|(-\Delta)^{\alpha/2} u\|_{L^2} + \|\rho^{-\alpha}(E - \delta)^{\alpha/2} u\|_{L^2} \leq c_2 \|(-\Delta)^{\alpha/2} u\|_{L^2} \]
where \( c_1, c_2 \) are constants which do not depend on the function \( u \) and by \( \Delta_\rho \) we have denoted the radial part of the Laplace operator \( \Delta \). With the aid of Parseval's equality and of the left-hand side of the inequality (3), the estimate of the norm of the operator \( A_{km} \) in the space \( L^2, \alpha \) reduces to the estimate of the norm \( \|\rho^{-\alpha}(E - \delta)^{\alpha/2} Y_{km} F_{\gamma - \xi} u\|_{L^2} \). But from Lemma 1 it follows that
\[ \sum_{m=0}^{\infty} \|\rho^{-\alpha}(E - \delta)^{\alpha/2} Y_{km} F_{\gamma - \xi} u\|_{L^2}^2 \leq C k^{n-2\alpha s}\|u\|_{L^2}^2. \]

Making use of the right-hand side of the inequality (3), we obtain the required estimate (2).

For \( \alpha < 0 \) we turn to the adjoint operator \( A^*_{km} \), which acts in the space \( L^2, -\alpha \), conjugate to \( L^2, \alpha \). It is easy to show that
\[ A^*_{km} = A_{km} - \sum_{|\alpha| > 0} D^{\alpha}_{\omega} A_{km}|_{x = x_{-}\omega}. \]

For the functions \( v \in Q_{-S} \) we have \( A^*_{km} v = A_{km} v \), and therefore
\[ (A_{km} u, v) = (u, A_{km} v), u \in Q_s, v \in Q_{-s}. \]

Making use of the first part of the proof for \( \alpha > 0 \), we obtain the required estimate, since
\[ \sum_{m=0}^{\infty} \|u, A_{km} v\|^2 \leq C k^{n-2\alpha |\alpha|}\|u\|^2, \|v\|^2. \]

For \( \alpha = 0 \), Lemma 2 follows at once from (1).

3. Let \( \Phi(x, \xi) \) be a positive function, homogeneous of degree zero with respect to \( \xi \). We associate to it the s.i.o.
\[ A = \sum_{\alpha=0}^{n} \sum_{m=0}^{\infty} a_{\alpha m} A_{\alpha m}, \]
where \( a_{\alpha m} \) are the coefficients of the expansion of the function \( \Phi(x, \xi) \) into a series of spherical functions \( Y_{km} \). We introduce the notation
\[ A_v = \sum_{\alpha=0}^{n} \sum_{m=0}^{\infty} a_{\alpha m} A_{\alpha m}. \]