AN INVARIANT NORMALIZATION OF A
SEMI-RIEMANNIAN MANIFOLD

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1. A semi-Riemannian manifold \((M, g)\) is said to be a differentiable manifold \(M\) where a field of a degenerate symmetric and twice covariant tensor \(g\) of constant rank is given. Another name for such manifolds is Riemannian manifolds with degenerate metric. Some problems of geometry of such manifolds were considered recently in [1-3].

A semi-Riemannian metric of rank \(r\) determines on the manifold \(M\) an isotropic distribution \(\Delta\), which is not involutive in general. A normalization of a semi-Riemannian manifold \((M, g)\) is said to be a distribution \(\Delta^*\) defined on \(M\) and complementary to \(\Delta\) [i.e., such that \(\Delta^*(x) + \Delta(x) = \mathbb{T}_x, \Delta^*(x) \cap \Delta(x) = \{x\}\), where \(x\) is a point of \(M\) and \(\mathbb{T}_x\) is its tangent space].

The main goal of the present paper is to construct an invariant normalization \(\Delta^*\) of a semi-Riemannian manifold \((M, g)\) which is intrinsically connected with the tensor field \(g\) given on \(M\). Conditions are found under which this construction is possible. In addition, an affine connection with torsion intrinsically connected with the tensor \(g\) is constructed in the paper. The distributions \(\Delta\) and \(\Delta^*\) turn out to be parallel with respect to this connection.

When we construct the invariant normalization \(\Delta^*\) and the affine connection mentioned, we suppose that the isotropic distribution \(\Delta\) is not involutive and essentially use its tensor of nonholonomicity. In this connection we apply the method developed by Laptev in [4]. In the case where \(\Delta\) is involutive, it is possible to find another method of construction of an invariant normalization. Such a construction was done in [3] for the case where the rank of the tensor \(g\) is one less than the dimension of the manifold \(M\).

2. Consider the fiber of linear frames \(L(M)\) connected with an \(n\)-dimensional differentiable manifold \(M\). Let \(\theta = \{\theta_i^j\}\) be the canonical 1-form of \(L(M)\) and \(\omega = \{\omega_i^j\}\) be a fiber 1-form of this fiber. These forms satisfy the structure equations

\[
d\theta_i^j = \theta_\ell^j \wedge \omega_i^\ell, \tag{1}
\]

\[
d\omega_i^j = \omega_k^i \wedge \omega_k^j + \theta_\ell^j \wedge \omega_i^\ell, i, j, k = 1, \ldots, n, \tag{2}
\]

where the forms \(\omega_{ijk}\) satisfy the relations

\[
\omega_{jk} \wedge \theta_i \wedge \theta^k = 0
\]

(see, for example, [5]). If one fixes a point \(x\) on the manifold \(M\), then the values of forms \(\theta_i^j\) become zero and Eqs. (2) take the form

\[
\delta \pi_i^j = \pi_k^j \wedge \pi_i^k
\]

where \(\delta\) is the operator of differentiation with respect to parameters of the linear group \(GL(n)\) which determines transformations of frames of the first order connected with the point \(x\), and \(\pi_i^j = \omega_i^j(\delta)\) are invariant forms of this group.

Let a field of a symmetric tensor \(g = (g_{ij})\) be given on the manifold \(M\), and let the rank of \(g\) be constant and less than \(n\): \(\text{rank} (g_{ij}) = r, 0 < r < n\). The matrix of this tensor can be reduced to the form

\[
g = \begin{bmatrix} a_{ab} & 0 \\ 0 & 0 \end{bmatrix},
\]

where \(a, b = 1, \ldots, r; \det (g_{ab}) \neq 0\). According to what has been said above, the metric determined by such a tensor on \(M\) is called semi-Riemannian or a degenerate Riemannian metric. The metric quadratic form of the
The manifold $M$ has in this case the form

$$ds^2 = g_{ab} \theta^a \theta^b.$$  

(3)

It follows from this that the directions determined by the system of equations

$$\theta^a = 0$$  

(4)

are isotropic directions of this metric. Since we did not suppose that the metric (3) is positive semidefinite, then isotropic directions not satisfying Eqs. (4) may exist on the manifold $M$. We exclude such isotropic directions from consideration.

The tensor field $g = (g_{ij})$ given on the manifold $M$ is determined by the equation

$$\nabla g_{ij} = \delta g_{ij} - g_{ik} \pi^k_j - g_{jk} \pi^k_i = 0$$  

(see, for example, [6]). In our case this system of equations can be written in the form

$$\delta g_{ab} - \delta g_{cb}^a - \delta g_{cb}^b = 0,$$

$$\delta g_{ab}^b = 0.$$  

(5)

Let us suppose that here and further the indices take the following values: $a, b, c, e = 1, \ldots, r; u, v, w, x, z, s, t = r + 1, \ldots, n$. It follows from the last equation that $\pi^a_u = 0$. These equations show that the group of admissible transformations of frames of the first order contracts because of the canonization made above, and the matrix of its invariant forms takes the form

$$\begin{pmatrix}
\pi^b_a & \pi^b_u \\
0 & \pi^b_u
\end{pmatrix}.$$  

It follows from Eqs. (5) that on the manifold $M$ the tensor $g_{ab}$ satisfies the equations

$$dg_{ab} = g_{ac} \omega^c_b - g_{bc} \omega^c_a = \bar{g}_{ab} \theta^a + \bar{g}_{ab} \theta^b,$$  

(6)

and forms $\omega^a_b$ are expressed in the form

$$\omega^a_b = \gamma^a_b \theta^b + \gamma^a_b \theta^a.$$  

(7)

Consider on the manifold $M$ the isotropic distribution $A$ determined by the system of equations (4). The dimension of this distribution is equal to $n - r$. Differentiating the left members of Eqs. (4) and using relations (7), we obtain

$$d\theta^a = \theta^b \wedge (\omega^b_s - \bar{\gamma}^b_s \theta^a) + \lambda^a_{uv} \theta^u \wedge \theta^v,$$  

(8)

where $\lambda^a_{uv} = \lambda^a_{[uv]}$. It is clear from this that the distribution $A$ is involutive if and only if $\lambda^a_{uv} = 0$.

We introduce on the fiber of frames $L(M)$ the new forms

$$\varphi^a_b = \omega^a_b - \bar{\gamma}^a_b \theta^b + \gamma^a_b \theta^a,$$

where $\gamma^a_bc = \gamma^a_{cb}$. Substituting these forms in relations (8), we obtain

$$d\theta^a = \theta^b \wedge \varphi^b_a + \theta^u \theta^v \wedge \theta^a.$$  

(9)

We express the forms $\omega^a_b$ in terms of $\varphi^a_b$ from previous relations and insert them in Eqs. (6). Following [6], one can show that coefficients of forms $\omega^a_b$ in these relations can be reduced to zero by an approximate choice of the quantities $\gamma^a_bc$. After this Eqs. (6) take the form

$$dg_{ab} - g_{ac} \varphi^c_b - g_{bc} \varphi^c_a = \bar{g}_{ab} \theta^a.$$  

(10)

3. For the construction of an invariant normalization of a semi-Riemannian manifold $M$ we need to find differential extensions of Eqs. (9) and (10). Taking exterior derivatives of Eqs. (9), we obtain

$$- \theta^b \wedge (dq^a_b - q^a_b \wedge \phi^a_b) + \nabla \lambda^a_{uv} \wedge \theta^u \wedge \theta^v - 2 \lambda^a_{uv} \theta^u \wedge \theta^v = 0,$$  

(11)

where

$$\nabla \lambda^a_{uv} = d\lambda^a_{uv} + \lambda^a_{uv} \omega^b_u - \lambda^a_{uv} \omega^b_u - \lambda^a_{uv} \omega^b_u.$$  

Forms $\omega^a_b$ satisfy equations similar to (2). Therefore

$$dq^a_b - q^a_b \wedge \phi^a_b = \theta^f \Lambda \phi^a_f + \theta^u \wedge \phi^a_u.$$  

(12)