A SYMMETRIC MAPPING OF SPATIAL DOMAINS,
INFINITELY CLOSE TO A SPHERE, WITH
ASYMPTOTICALLY SMALLEST MAXIMAL DILATATION

M. Yu. Vasil'chik

1. INTRODUCTION

1.1. Let us consider two domains $U$ and $V$ in the $n$-dimensional arithmatical Euclidean space $\mathbb{R}^n$ ($n \geq 3$). Let us denote the set of all quasiconformal mappings of the domain $U$ onto the domain $V$ by $\mathcal{H}(U, V)$. Suppose that $K$ is some fixed maximal dilatation. If $\mathcal{H}(U, V) \neq \emptyset$, then the quantity

$$\delta_K(U, V) = \inf \{K(f); f \in \mathcal{H}(U, V)\}.$$

is defined.

The quantity $\delta_K(U, V)$ is called the smallest maximal dilatation of the ordered pair of domains $(U, V)$. A mapping $f$ from $\mathcal{H}(U, V)$ is said to be extremal if

$$K(f) = \delta_K(U, V).$$

1.2. Let $\Pi$ be a subgroup of the group of all conformal automorphisms of the space $\mathbb{R}^n$, where the completion of $\mathbb{R}^n$ by a single point at infinity. A domain $U \subset \mathbb{R}^n$ is said to be $\Pi$-symmetric if $P(U) = U$ for all $P \in \Pi$.

If a domain $U$ is $\Pi$-symmetric, then a mapping $f: U \rightarrow \mathbb{R}^n$ is said to be $\Pi$-symmetric if

$$P(f(x)) = f(P(x)) \quad (1.1)$$

for all $x \in U$ and all $P \in \Pi$.

1.3. Let the domains $U$ and $V$ be $\Pi$-symmetric with respect to a subgroup $\Pi$, $\mathcal{H}(U, V) \neq \emptyset$ and $\delta_K(U, V) > 1$. Do the extremal mappings from $\mathcal{H}(U, V)$ contain a $\Pi$-symmetric mapping among them? Although this problem is known for quite long in the theory of quasiconformal mappings, yet it has not been solved till now even for particular cases of pairs of domains and prime subgroups $\Pi$. In particular, no results about this problem are known in the case where one of the domains is the unit ball and the group of rotations of the space is considered as the subgroup $\Pi$.

In the present article, we consider the problem about the existence of a $\Pi$-symmetric extremal mapping in the case of a one-parameter family of $\Pi$-symmetric domains $\{B_t\}$ that contracts regularly to the unit ball $B$ as $t \rightarrow 0$ (the exact definition is given below (see Sec. 1.4)) when the subgroup $\Pi$ is the group of rotations of the space $\mathbb{R}^n$ that leaves fixed the points of the $k$-dimensional subspace $I_k$, $1 \leq k \leq n - 2$. 

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For \( k = 1 \) the subgroup \( \mathcal{H} \) is the group of rotations of \( \mathbb{R}^n \) about the first coordinate axis. In the case under consideration, the problem of existence of a \( \mathcal{H} \)-symmetric extremal mapping remains open, but it can be shown that for all sufficiently small \( t \) the set \( \mathcal{H}(B, B_t) \) contains a \( \mathcal{H} \)-symmetric mapping whose maximal dilatation has the same asymptotic for \( t \to 0 \) as the smallest maximal dilatation.

1.4. Before stating the obtained result precisely, we give some definitions.

A family of domains \( \{B_t\}, t \in [0, \delta), \) contracts regularly to a ball \( B \) as \( t \to 0 \) if \( B_0 = B \) and there exists a mapping \( \Phi: [0, \sigma_B] \times B \to \mathbb{R}^n, \sigma_B > 0, [0, \sigma_B] \subset [0, \delta) \) that satisfies the following conditions:

\[
\begin{align*}
\Phi(0, x) &= x, \quad x \in B; \\
\Phi(t, B_t) &= B_t, \quad t \in [0, \sigma_B]; \\
\Phi &\in C^1([0, \sigma_B] \times B).
\end{align*}
\]

We will denote the boundary of the unit ball \( B = \{x \in \mathbb{R}^n : |x| < 1\} \) by \( S \). Let us set

\[
\hat{\psi}(t, x) = \psi(x); \quad (\psi(s), s) = \varphi(s), \quad s \in S.
\]

The function \( \varphi \) does not depend on the choice of the mapping, satisfying conditions (1.2)-(1.4). This is easily verified if an arbitrary point \( s \in S \) is fixed and the surface \( \partial B_t \) is given in a certain spatial neighborhood of \( s \) implicitly by the equations

\[
W(t, z) = 0, \quad z \in \mathbb{R}^n, \quad t \in [0, \sigma_B],
\]

where the function \( W \) is continuously differentiable with respect to \( t \) and \( z \). Let us observe that here we can assume that the mapping \( \Phi \) belongs to the space \( C^1([0, \sigma_B] \times \overline{B}) \). The function \( \varphi \) is called the initial boundary function of the family \( \{B_t\} \).

1.5. We show that there exists a mapping \( \psi_0(t, s) \) of class \( C^1([0, \sigma_B] \times \overline{B}) \) that is \( \mathcal{H} \)-symmetric and has the form

\[
\psi_0(t, s) = \psi(1 + t\varphi(s) + t\varepsilon_0(s))
\]

and satisfies the conditions (1.2) and (1.3). At first, let us construct such a mapping on \( S \). Let \( \psi_0(t, s) \) denote the point of intersection of the ray that issues from the origin and passes through \( s \in S \) with \( \partial B_t \) — the boundary of the domain \( B_t \). For small \( t \), the point \( \psi_0(t, s) \) is defined uniquely. Thus, we obtain a mapping \( s \to \psi_0(t, s) \) that maps \( S \) onto \( \partial B_t \). It follows from the definition of regular contractibility that the mapping \( \psi_0(t, s) \) is a \( C^1 \)-diffeomorphism of \( S \) onto \( \partial B_t \). Moreover, it can be easily verified that the mapping \( t \to \psi_0(t, s) \) is continuously differentiable with respect to \( t \) and

\[
\left( \frac{\partial \psi_0}{\partial t}(0, s), s \right) = \varphi(s), \quad s \in S.
\]

Let us set \( \varepsilon_0(t, s) = t^{-1} \left( \psi_0(t, s) - s - t\varphi(s), s \right) \). Then \( \varepsilon_0 \in C^1(S) \) and \( \|\varepsilon_0\|_{C^1(S)} \to 0 \) as \( t \to 0 \). (See, e.g., [1] for the definition of the space \( C^1(S) \) and the norm in it.)

It follows from the construction and the \( \mathcal{H} \)-symmetry of \( B \) and \( B_t \) that \( \psi_0(t, s) \) is a \( \mathcal{H} \)-symmetric mapping. The mapping \( \varepsilon_0(t, s) \) has the form (1.5) on \( S \). Since \( \psi_0(t, s) \) is \( \mathcal{H} \)-symmetric, it follows that for the functions \( \varphi \) and \( \varepsilon_0 \) we have

\[
\varphi(s) = \varphi(P_s), \quad \varepsilon_0(s) = \varepsilon_0(P_s)
\]

for all \( P \in \mathcal{H} \) and all \( s \in S \). We will denote the projection of the point \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) on \( L_k \) by \( x' \). Equations (1.6) mean that the functions \( \varphi \) and \( \varepsilon_0 \) depend actually only on the first \( k \) coordinates of the point \( s = (s_1, \ldots, s_n) \), i.e., on \( s' \). Thus, \( \varphi(s) = \varphi(s') \), \( \varepsilon_0(s) = \varepsilon_0(s') \). Let us set \( \hat{\psi}(x) = \varphi(x') \), \( \varepsilon_0(x) = \varepsilon_0(x') \) for \( x \in B \). Then \( \varphi \) and \( \varepsilon_0 \) belong to \( C^1(\overline{B}) \) and \( \|\varepsilon_0\|_{C^1(\overline{B})} \to 0 \) as \( t \to 0 \). We set

\[
\psi_0(t, x) = x(1 + t\varphi(x) + t\varepsilon_0(x))
\]

for \( x \in B \). The mapping \( \psi_0 \) satisfies conditions (1.2), (1.3), and (1.5), belongs to \( C^1([0, \sigma_B] \times \overline{B}) \), and is \( \mathcal{H} \)-symmetric.

1.6. Let us define the maximal dilatation for spatial mappings. We restrict ourselves only to four maximal dilatations, most useful in the investigations about quasiconformal spatial mappings.

Let \( U \) be a domain in \( \mathbb{R}^n \) and \( f: U \to \mathbb{R}^n \) be a mapping. The mapping \( f \) is said to be quasiconformal if it is homeomorphic and belongs to the space \( W_{1,1}^{1,1}(U) \) and one of the following quantities is finite for it: