


**Inverse Scattering Problems and Restoration of a Function from the Modulus of Its Fourier Transform**

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**INTRODUCTION**

The following inverse scattering problem is studied: to find the shape of the surface, given the intensity of the coherent monochromatic light scattered by it in the far zone — the Fresnel diffraction zone or Fraunhofer diffraction zone [1] (for other statements of the inverse scattering problem, see [2-7]).

Mathematically, these problems reduce to the following: assume that $\Omega$ is a bounded domain in $\mathbb{R}^n$, $f \in C(\Omega)$ is a complex-valued function,

$$g(x) = \left| \int_{\Omega} \exp[-i \langle x, \xi \rangle] f(\xi) d\xi \right|^2, \ x \in \mathbb{R}^n.$$  \hspace{1cm} (0.1)

To restore function $f$, given function $g$.

Though problem (0.1) has been considered earlier (see [8-16] and the literature cited there), the uniqueness theorems seem only to have been strictly proved in the one-dimensional case; and it is assumed in these theorems either that $f$ is an analytic function [9-11, 16], or that the diameter of $\Omega$ is sufficiently small [12, 13]; in [8], $\Omega = (-1, 1)$, and $f$ is an even real function; in inverse scattering problems it is usually assumed that function $f$ is complex-valued (see (1.1)-(1.4), [14]).

In the present article uniqueness theorems are proved for problem (0.1) in the class $f \in C^\omega(\Omega)$ (the smoothness can be reduced, see Note 2.1). The results of the article were announced in [17] (a lemma in [17] and Theorems 1-3 were proved by the present author). All the functions considered below are complex-valued unless otherwise stipulated.

**1. Statements of Inverse Scattering Problems**

The statements, cited here are due to V. G. Volostnikov and V. V. Kotlyar and are published here with their kind permission.

Let the plane $\{x_3 = 0\}$ in $\mathbb{R}^3$ be filled by a black screen with a hole $\Omega$, where $\Omega$ is a bounded domain in $\mathbb{R}^2$. Let the body $T \subset \mathbb{R}^3$ be such that

$$T = \{x| (x, x_3) \in \Omega, \ x_3 \in (0, p(x, x_3))\},$$

$$p(x, x_3) \in C(\Omega), \ p > 0 \ in \Omega.$$

We assume that a plane monochromatic wave with wave number \( k \), \( u_0 = \exp(ikx_3) \) (laser beam), is incident on \( T \) from half-space \( \{ x_3 < 0 \} \). Let \( T \) be a "thin phase transparency," i.e., on passing through \( T \), only the wave (light) phase changes, the angular deviation of the scattered rays from \( x_3 \) axis being small [18, p. 320]. Then the amplitude of the refracted wave (we ignore reflection) immediately behind \( T \) has the form [19, p. 72]

\[
\begin{align*}
u = \exp(ik\psi(x_1, x_2)), \\
\psi(x_1, x_2) = n(x_1, x_2)p(x_1, x_2),
\end{align*}
\]

where \( n(x_1, x_2) \) is the refractive index of \( T \). By the Fresnel–Kirchhoff principle, with \( k, x_3 \gg 1 \) [1, p. 249],

\[
u(x) = -\frac{ik}{2\pi r} \int_0^1\exp(ik\psi(\xi))d\xi,
\]

\[
r = \sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + x_3^2}.
\]

Let \( P \) be a finite domain in \( \mathbb{R}^2 \) and \( k \gg 1 \), \( \frac{|x_3 - \xi_3|}{|x_3|} \ll 1 \) \( V(x_1, x_2) \subseteq P, V(\xi_1, \xi_2) \subseteq \Omega \), i.e., \( \{ x | (x_1, x_2) \in P, x_3 = \text{const} \} \) is the "brightness domain" on the plane \( \{ x_3 = \text{const} \} \). Then, in the Fresnel approximation, for these \( x_3 \) we can put [1, p. 255]

\[
u(x_1, x_2, x_3) = -\frac{ik}{2\pi x_3}\exp(ikx_3)\int_0^1 \exp \left[ ik \left( \frac{|x - \xi|^2}{2x_3} \right) \right] \exp(ik\psi(\xi))d\xi,
\]

where \( \xi = (x_1, x_2) \).

**Problem 1.1.** We know the functions

\[
H_j(x) = \left| \int_0^1 \exp \left[ ik \left( \frac{|x - \xi|^2}{2x_3} \right) \right] f(\xi) d\xi \right|^2.
\]

Here, \( x = (x_1, x_2) \in P, j = 1, 2, \alpha_1, \alpha_2 = \text{const} \in R, \alpha_1, \alpha_2 \neq 0, \alpha_1 \neq \alpha_2 \). We want to find function \( f \in C(\Omega) \).

In other words, \( \{ x_3 = \alpha_j \} \) is the plane in \( \mathbb{R}^3 \) on which the scattered light intensity is measured. Since functions \( H_j \) are analytic in \( \mathbb{R}^2 \), being functions of real variables, and are known in the domain \( P \subset \mathbb{R}^2 \), they can be assumed to be known for all \( x \in \mathbb{R}^2 \).

Problem (1.1) can also be interpreted as the problem of finding the initial condition in the nonstationary Schrödinger equation from measurements of the modulus-squared of its solution at different instants.

Now let \((\xi_1^2 + \xi_2^2)(2x_3)^{-1} \ll 0\) in (1.1) be the Fraunhofer zone [1, p. 259]. Denoting \( y_j = -kx_j(2x_3)^{-1} \), we obtain

\[
Q(y) = \left| \int_0^1 \exp \left[ i \langle y, \xi \rangle \right] \exp(ik\psi(\xi)) d\xi \right|^2,
\]

where \( Q(y) = |u2\pi x_3 k^{-1}|^2 \).

**Problem 1.2.** Function \( Q(y), y \in \mathbb{R}^2 \), is known. To find function \( \psi \).

If we consider laser light scattering at a convex body \( T \), we can use arguments of the type of [7] to obtain for the scattered light intensity \( H_j(x) \), measured on the plane \( \{ x_3 = \alpha_j \} \), in the Fresnel–Kirchhoff approximation the expression (1.2) with

\[
f(\xi) = \exp(ikp(\xi)),
\]

where \( \{ x_3 = p(x_1, x_7) \} \) is the equation of the illuminated part \( S \) of surface \( T, \Omega \) is the orthogonal projection of \( S \) onto plane \( \{ x_3 = 0 \} \). The situation (1.4) is encountered, e.g., when analyzing "roughnesses" of a surface, due to finishing defects.

2. Some Lemmas

We shall prove some lemmas, needed for what follows.